Redundancy-Aware Electromigration Checking for Mesh Power Grids[*]

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ABSTRACT

Electromigration (EM) is re-emerging as a significant problem in modern integrated circuits (IC). Especially in power grids, due to shrinking wire widths and increasing current densities, there is little or no margin left between the predicted EM stress and that allowed by the EM design rules. *Statistical Electromigration Budgeting* (SEB) estimates the reliability of the grid by considering it entirely as a *series system*. However, a power grid with its many parallel paths has much inherent redundancy. In this paper, we propose a new model to estimate the MTF and reliability of the power grid under the influence of EM, which accounts for these redundancies. We refer to this as the *mesh* model. To implement the *mesh* model, we also develop a framework to estimate the change in statistics of an interconnect as its *effective-EM* current varies. The proposed algorithm is quite fast and has an overall *observed empirical* complexity of $O(n^{1.4})$. The results indicate that the *series* model, which is currently used in the industry, gives a pessimistic estimate of power grid MTF and reliability by a factor of 3-4.

Keywords

Power grid, Electromigration, MTF Estimation, Verification, Redundancy

1. INTRODUCTION

Verifying the power grid is a crucial step in VLSI design, as the reliability of the underlying logic heavily depends on its power grid. Not only must an IC perform as desired, it must also survive and function as intended for several years before failing. With the complexity of modern designs, reliability is becoming a more serious concern. Specifically, electromigration (EM) has re-emerged as a significant problem in modern chip design and there are three problems that demand attention: 1) existing EM checking techniques for the power grid are overly pessimistic (because of an underlying *series system assumption*, as we will explain), leading to loss of safety margins and multiple design iterations, 2) increased current density in grid metal lines has led to a significant loss of margins between the predicted EM stress and the allowed thresholds, and 3) checking modern, large power grids for EM has become very expensive. To make things worse, it is forecast [18] that metal line current density and reliability due to EM will get dramatically worse with continued technology scaling. As a result, EM signoff has become increasingly difficult and designers are forced to reconsider traditional approaches, and to look with suspicion at the large safety margins and pessimism built into traditional EM checking methods.

Historically, 'worst case' current density limits for individual lines were used to arrive at reliable designs for ICs. However, this approach severely restricted the design process, which motivated the need for a model to relate the reliability of individual components to the reliability of the entire system. In an early contribution [6] a *series model* was proposed to determine the reliability of an IC, under which a system is deemed to have failed as soon as any of its components fails. Under the series model, and with some simplifying assumptions, the failure rate of the system is the sum of failure rates of individual components. The series model was applied to the Alpha 21164 microprocessor, under the name *Statistical Electromigration Budgeting* (SEB) [9] and became a standard technique in many industrial CAD tools.

However, modern power grids use a mesh structure. With their many paths for current flow, meshes have much redundancy and are in fact closer to (but not quite) a parallel system, rather than a series system, and so have a longer lifetime than a series system. This issue has largely been ignored in EM checking tools, both in academia and industry; no industrial tool has this feature today. While SEB accounts for the fact that EM failures are statistical it does not, however, recognize the benefits of redundancy and it treats the overall metal structure as a series structure. A power grid is not necessarily failed if one of its metal lines fails. Instead, in our approach, we deem a power grid to have failed only when enough lines have failed that the voltage on the grid becomes unacceptable. Our data shows that, in many cases, a grid can tolerate up to 30 or more line failures before it truly fails! In this work, we develop a more realistic grid EM checking and budgeting method that takes the redundancy of the grid structure into account. Specifically, a) we develop a new model, referred as the *mesh* model, that intelligently factors in the redundancy of the power grid while estimating its MTF and reliability and b) we propose a novel framework to estimate the change in statistics of an interconnect as its *effective-EM* current varies in steps on a time-scale comparable to the failure times of interconnects, as is the case in *mesh* model. In implementing the mesh model, we also develop an efficient *exact* method to update the node voltage drops as the structure of the grid changes due to failure of interconnects. Our preliminary analysis using publicly available grids from IBM, with up to 700k nodes, and internally generated grids, with up to 1 million nodes, show an increase in predicted lifetime of 3-4x compared to the existing series-system based approach. A lot of margin is therefore "left on the table" and there is room for high-impact improvements in EM verification.

The remainder of the paper is organized as follows. In section 2, we present a background on EM and the power grid model. Section 3 describes the *mesh* model and the proposed framework for computing the statistics of a metal line in the scenario of changing currents. Implementation details are provided in section 4 followed by the experimental results in section 5. Finally section 6 concludes the paper.

2. BACKGROUND

2.1 Electromigration

Electromigration is the mass transport of metal due to momentum transfer between electrons (driven by an electric field) and diffusing metal atoms. Failure occurs in metal lines only when there is a *flux divergence* with regard to movement of metal

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[*] This work was supported in part by the Natural Sciences and Engineering Research Council (NSERC) of Canada.

978-1-4799-1071-7/13/$31.00 ©2013 IEEE

atoms. The distortions in the lattice, in the form of vacancies and grain boundaries, allow for *diffusion* of metal atoms, which leads to *flux divergence* and hence failure due to EM.

2.1.1 EM model: Black's Equation

Under the influence of Electromigration, metal line resistance increases as a line approaches failure and starts to deform due to void creation or hillock formation. Empirically, Black's Equation has been used to determine the *Mean Time to Failure* or MTF of metal line under the influence of electromigration [2]:

$$MTF = \frac{wt}{A} J^{-n} \exp\left(\frac{E_a}{kT_m}\right) \tag{1}$$

where A is an empirical constant that depends on a host of physical properties such as volume resistivity of the metal and the effective ionic scattering cross section for electrons; w and t are the width and thickness of the line respectively, J is the *effective-EM* current density, n is a current exponent, k is the Boltzmann's constant, T_m is the temperature in Kelvin and E_a is the activation energy for Electromigration. A current density exponent $n = 1$ is consistent with void growth limited failure [8] and $n = 2$ indicates void nucleation limited failure [10].

2.1.2 Blech Effect

If the line is short enough, then the back-stress developed due to accumulation of atoms at the ends of the line could overcome the build-up of the critical stress required for creation of a void in a line, and thus the line is no longer susceptible to EM failure [3]. This effect, called the *Blech effect*, is quantified in terms of a critical value of product between current density (J) and length of a line (L), denoted by β_c. For a given current density J, a line is said to be *EM-immune* if $JL \leq \beta_c$ and *EM-susceptible* if $JL > \beta_c$.

2.1.3 Lognormal distribution

Under identical conditions of geometry, current and temperature, the rate of EM degradation depends on the specific microstructure of the metal line. As a result, due to random manufacturing variations, the Time-to-Failure (TTF) is a random variable $\mathbf{T}$, typically modeled with a *lognormal distribution* with the mean time-to-failure (MTF) as given by (1). The deviation of $\ln \mathbf{T}$, denoted by σ_{ln}, is dependent on the ratio of width to median grain size of the metal line for aluminum [4] and on the product JL^2 for copper [12], and this dependence is determined experimentally. However, the deviation in both cases eventually levels off to a constant value. Hence, in this paper we assume that σ_{ln} is constant for a given material, as is typical in the literature.

2.1.4 Obtaining a TTF sample

Given that the Time-To-Failure of an interconnect in a power grid has a lognormal distribution, a TTF sample from the distribution can be obtained using:

$$\tau = \exp\left(\mu_{\ln} + \Psi\sigma_{\ln}\right) \tag{2}$$

where $\mu_{\ln} = E[\ln \mathbf{T}]$ and Ψ is a sample value from *standard normal distribution* Φ. We know that for a lognormal distribution:

$$\mu_{\ln} = \ln(\mu_T) - 0.5\sigma_{\ln}^2 \tag{3}$$

where $\mu_T = E[\mathbf{T}]$ and is empirically determined using (1). From (3) and (2), we finally get:

$$\tau = \exp\left(\ln(\mu_T) - 0.5\sigma_{\ln}^2 + \Psi\sigma_{\ln}\right) = \mu_T \exp\left(\Psi\sigma_{\ln} - 0.5\sigma_{\ln}^2\right) \tag{4}$$

2.1.5 Effective-EM current

Because EM is a long-term cumulative failure mechanism, the changes in line current on very short time-scales (such as the normal operation of a digital chip), are not terribly significant. Instead, the standard approach is to compute an *effective-EM* current, which is a constant current value, derived from the line current waveform which gives the same lifetime for that line under the influence of electromigration. In traditional EM work, this effective current is computed based on some assumed periodic current waveform. If the waveform is uni-directional, then *Direct Current EM analysis* is used, based on time-averaged current density (J_{avg}). For more general cases of bi-polar currents, *transient current EM analysis* uses an *effective-EM* current density of [11]:

$$J_{tran,EM.eff} = \frac{1}{T}\left(\int_0^T J^+(t)dt - \varphi \int_0^T |J^-(t)|dt\right) \tag{5}$$

where $\int J^+(t)dt$ is the integral of one side of the current waveform and is larger than the opposite side (i.e. $\int J^-(t)dt$) and φ is the EM recovery factor, that is determined experimentally.

2.2 Power Grid Model

The power grid nodes are numbered $1, 2 \ldots m$, with the ground node being 0. The currents drawn by the underlying logic blocks are modeled by a set of current sources connected between a subset of the grid nodes and the ground. An interconnect in the power grid mostly carries uni-directional currents, so that its EM analysis (using Black's Equation) depends on the *average* current only. Because the power grid is a linear system, the average current in an interconnect can be obtained directly from the circuit current averages by doing a DC analysis. The same DC analysis also gives us the average node voltage drops. In the next section, we propose an EM verification framework that depends on user-provided thresholds on the average voltage drops. In this framework, it becomes sufficient therefore to perform a DC analysis of the grid driven by the averages of the circuit currents. Using Modified Nodal Analysis, the equation modeling a DC power grid can be written as:

$$G(t)v(t) = i \tag{6}$$

where $G(t)$ is the $m \times m$ conductance matrix (it varies over time, *over large time-scales*, as the lines age and deform, hence the time dependence), $v(t)$ is the corresponding vector of node-voltage drops and i is the vector of source average currents tied to the grid (which model the underlying logic circuits).

3. PROPOSED APPROACH

As our first contribution in this paper, we propose a new model to determine the failure of the grid taking into account the redundancies due to it's mesh structure. Note that the development of the *mesh* model in section 3.1, and the material in section 3.2 upto eq. (13), overlaps with [5]. This small overlap is unavoidable due to lack of space and to make sure each paper is understandable on its own.

3.1 New grid failure model: The mesh model

Circuit timing is dependent on the grid voltage drop [1]. Hence, the *safety* of a power grid is dictated by the *safety* of its nodes. A node is said to be *safe* when its voltage drop is below a user-specified *threshold*. We define a power grid to be *safe* if all its nodes are *safe*, otherwise it is deemed to have *failed*. Since the voltage drop of a node is determined from the grid topology itself, this model automatically captures the redundancies in the grid.

We assume that at $t = 0$, the grid is *connected*, so that there is a resistive path from any node to any other node that does not go through a v_{dd} supply node or a ground node. A *voltage-drop threshold* value for every grid node (or a subset of grid nodes) is given, and is captured in the vector v_{th}. We assume that $i \neq 0$ and $v_{th} > 0$, to avoid the trivial cases. Also, we assume that the grid is initially safe so that all its voltage drops are below user-specified thresholds, i.e., if $v_0 = v(0)$, then we have $v_0 < v_{th}$. As we move forward in time, the grid lines start to *fail* due to electromigration. We define failure of an interconnect to be an *open circuit*. The reader should note that this is a *conservative* approach because typical EM models assume a line to have failed once its resistance has risen above some threshold. Thus, lines continue to conduct current, albeit with high resistance, after the time-to-failure (TTF) predicted by typical EM models. By assuming infinite resistance after the predicted TTF, our approach is thus conservative. Hence, every failure corresponds to removal

```
while (Monte Carlo has not converged)
    Use a random number generator to generate TTF
    samples for all interconnects in the power grid
    while (grid has not failed)
        Remove the surviving line with lowest TTF
        Update all node voltage drops
        Update statistics of surviving lines
    end while
    Update the power grid MTF estimate
end while
```

Figure 1: Pseudo-code for *mesh* model

of a resistor from the grid, increasing grid sparsity. Thus, $\|G(t)\|$ decreases with time, which leads to an increase in $\|v(t)\|$:

$$\|G(t)\|\|v(t)\| \geq \|G(t)v(t)\| = \|i\| \implies \|v(t)\| \geq \frac{\|i\|}{\|G(t)\|} \qquad (7)$$

In this model, the grid is deemed to fail at the earliest time for which the condition $v(t) \leq v_{th}$ is no longer true, which can happen either due to voltage drop(s) at some node(s) exceeding their threshold value or due to a *singular* grid (i.e. as the resistors are removed, a node becomes completely disconnected from the other nodes, causing $\det(G(t)) = 0$). A singular grid models the scenario where all resistors connected to a particular node have higher than threshold resistance due to EM. As a conservative approximation we assume this node has failed, causing the grid to fail. Once a grid has failed, it is assumed that it remains failed for all future time. This model, used to determine the Time-to-Failure of the grid, is henceforth referred to as the *mesh* model.

In order to estimate the MTF and reliability of the power grid using the *mesh* model, we perform *Monte Carlo* analysis, as shown in Fig 1. Ideally, after each interconnect failure, we can simply find the new voltage drops by solving $v(t) = G^{-1}(t)i$ using LU factorization. However, this is too expensive. In the next section, we describe a highly efficient approach to update the voltage drops as interconnects fail in the power grid.

3.2 Efficiently updating the voltage drops

In order to update the voltage drops efficiently, we observe that G undergoes a *rank*-1 update ΔG_k after the k^{th} interconnect fails, where ΔG_k is the the conductance stamp of the k^{th} interconnect in the MNA matrix [15]. Any $m \times m$ *rank*-1 matrix can be written as the outer product of two column vectors of size m. Suppose g_k is the conductance of the k^{th} failed interconnect, connected between nodes p and q with $p > q$. Then, $\Delta G_k = u_k h_k^T$, such that

$$u_k = -h_k = \sqrt{g_k}\,(e_p - e_q) \qquad (8)$$

where e_λ is a column vector of size m containing 1 at the λ^{th} location and zeroes at all other locations, with e_0 being a vector of all zeros. Define U so that after k failures, $U \triangleq [u_1 \quad \ldots \quad u_k]$ and $H = -U$. Clearly, $\sum_{j=1}^{k} \Delta G_k = UH^T$. Thus, we can write the vector of voltage drops v_k after k interconnect failures as:

$$v_k = \left(G_0 + \textstyle\sum_{j=1}^{k} \Delta G_k\right)^{-1} i = \left(G_0 + UH^T\right)^{-1} i \qquad (9)$$

where G_0 is the conductance matrix at $t = 0$. Now, the *Woodbury* formula gives [7]:

$$\left(G_0 + UH^T\right)^{-1} = G_0^{-1} - \left[G_0^{-1}U(I_k + H^T G_0^{-1}U)^{-1}H^T G_0^{-1}\right] \qquad (10)$$

where I_k is $k \times k$ identity matrix. Using (10) in (9), we have:

$$v_k = G_0^{-1}i - \left[G_0^{-1}U(I_k + H^T G_0^{-1}U)^{-1}H^T G_0^{-1}\right] i \qquad (11)$$

We know that $G_0^{-1}i = v_0$, where v_0 is the voltage drop vector at $t = 0$. Define $Z \triangleq G_0^{-1}U = [G_0^{-1}u_1 \quad \ldots \quad G_0^{-1}u_k]$. Both v_0 and Z can be efficiently found using one sparse *LU* factorization of G_0 and $k + 1$ forward/backward substitutions (using triangular matrices L_G and U_G). Hence, we can re-write (11) as:

$$v_k = v_0 - Z(I_k + H^T Z)^{-1} H^T v_0 \qquad (12)$$

Let $W_k \triangleq (I_k + H^T Z)$ and $y_k \triangleq H^T v_0$. Eq. (12) becomes:

$$v_k = v_0 - Z W_k^{-1} y_k \qquad (13)$$

Even though the matrix W_k is dense, the number of interconnects k required to fail a grid is a very small fraction of the number of nodes m, hence finding $W_k^{-1}y_k$ by LU factorization is comparatively cheap. The matrices Z and $H^T Z$ need not be calculated from scratch for each failure, but can be efficiently updated by appending appropriate vectors at the end. However, for large grids, the number of interconnect failures required to fail a grid can become large. Solving a dense linear system $W_k^{-1}y_k$ using *LU* factorization has $O(k^3)$ complexity. In effect, this means that as k increases, the voltage updates using *Woodbury formula* slows down. To overcome this limitation, we propose a further refinement based on *Banachiewicz-Schur form* so that the complexity is reduced to $O(k^2)$.

We observe that W_k can be written as:

$$\begin{aligned} W_k &= I_k + H^T Z = I_k - U^T G_0^{-1} U \\ &= \begin{bmatrix} 1 - u_1^T G_0^{-1} u_1 & -u_1^T G_0^{-1} u_2 & \ldots & -u_1^T G_0^{-1} u_k \\ -u_2^T G_0^{-1} u_1 & 1 - u_2^T G_0^{-1} u_2 & \ldots & -u_2^T G_0^{-1} u_k \\ \vdots & \vdots & \ddots & \vdots \\ -u_k^T G_0^{-1} u_1 & -u_k^T G_0^{-1} u_2 & \ldots & 1 - u_k^T G_0^{-1} u_k \end{bmatrix} \end{aligned} \qquad (14)$$

Since G_0^{-1} is symmetric, then $u_i^T G_0^{-1} u_j = u_j^T G_0^{-1} u_i \; \forall i, j$. Hence, W_k is symmetric and we can re-write W_k in terms of W_{k-1} as:

$$W_k = \begin{bmatrix} W_{k-1} & b_k \\ b_k^T & d_k \end{bmatrix} \qquad (15)$$

such that

$$\begin{aligned} b_k &= [-u_1^T G_0^{-1} u_k \quad \ldots \quad - u_{k-1}^T G_0^{-1} u_k]^T \in \mathbb{R}^{k-1} \\ d_k &= 1 - u_k^T G_0^{-1} u_k \in \mathbb{R} \end{aligned} \qquad (16)$$

Hence, using the *Banachiewicz-Schur form*,we can express W_k^{-1} in terms of W_{k-1}^{-1} (see Appendix):

$$W_k^{-1} = \begin{bmatrix} W_{k-1}^{-1} + \dfrac{W_{k-1}^{-1} b_k b_k^T W_{k-1}^{-1}}{s_k} & \dfrac{-W_{k-1}^{-1} b_k}{s_k} \\ \dfrac{-b_k^T W_{k-1}^{-1}}{s_k} & \dfrac{1}{s_k} \end{bmatrix} \qquad (17)$$

where s_k is the Schur complement of W_{k-1} in W_k:

$$s_k = d_k - b_k^T W_{k-1}^{-1} b_k \qquad (18)$$

Also, we know that after k interconnect failures, we can update y_k from y_{k-1} by appending $p_k \triangleq -u_k^T v_0$ at the end:

$$y_k = H^T v_0 = -[u_1 \quad \ldots \quad u_k]^T v_0 = [y_{k-1}^T \quad p_k]^T \qquad (19)$$

Then, we can write $W_k^{-1}y_k$ as:

$$W_k^{-1} y_k = \begin{bmatrix} W_{k-1}^{-1} y_{k-1} + \dfrac{W_{k-1}^{-1} b_k b_k^T W_{k-1}^{-1} y_{k-1}}{s_k} - \dfrac{W_{k-1}^{-1} b_k}{s_k} p_k \\ -\dfrac{b_k^T W_{k-1}^{-1} y_{k-1}}{s_k} + \dfrac{p_k}{s_k} \end{bmatrix} \qquad (20)$$

But, the previous solution $\gamma_{k-1} \triangleq W_{k-1}^{-1} y_{k-1}$ is known, therefore:

$$\gamma_k = \begin{bmatrix} \gamma_{k-1} + \dfrac{W_{k-1}^{-1} b_k b_k^T \gamma_{k-1}}{s_k} - \dfrac{W_{k-1}^{-1} b_k}{s_k} p_k \\ -\dfrac{b_k^T \gamma_{k-1}}{s_k} + \dfrac{p_k}{s_k} \end{bmatrix} \qquad (21)$$

Define $a_k = \dfrac{b_k^T \gamma_{k-1} - p_k}{s_k}$. Now, we can rewrite (21) as:

$$\gamma_k = \begin{bmatrix} \gamma_{k-1} + a_k W_{k-1}^{-1} b_k \\ -a_k \end{bmatrix} \tag{22}$$

Hence, we can use (17) and (22) to directly update W_k^{-1} and γ_k from their previous values. Notice that W_k^{-1} is required because, in the next iteration, $W_k^{-1} b_{k+1}$ is needed to compute γ_{k+1} using (22). The implementation requires a single matrix-vector product ($O(k^2)$) and $O(k^2)$ additions and divisions (see Algorithm 2). Woodbury-Banachiewicz-Schur formulation results in a significant speed-up, for example we obtain a speed-up of **10.83X** by using Woodbury-Banachiewicz-Schur formulation over Sparse LU for updating voltage drops of the grid G4 ($\sim 450k$ nodes) as its interconnects fail.

An interconnect failure in the power grid changes the currents through all the surviving interconnects and hence affects their residual lifetime. Since the sparsity of the grid increases due to the failure of interconnects and the vector of source currents i is constant, the surviving interconnects (on an average) should conduct higher currents, which makes them more susceptible to failure due to EM. Thus, the failure statistics of the surviving interconnects should be modified to reflect the same. If ignored, it leads to an optimistic estimate of grid TTF which is undesirable. As our second contribution, we develop a novel approach to estimate the change in failure statistics of an interconnect when its *effective-EM* current density changes over time.

3.3 Estimating EM statistics for step currents

3.3.1 Motivation - the Single step case

Consider a thought experiment in which a large set S_0 of N isolated conductors are tested for their failure times. The testing starts at $t = 0$. Let the current densities through all the conductors be identical and given by the following step function:

$$J(t) = \begin{cases} J_0, & 0 \leq t \leq t_1 \\ J_1, & t_1 < t < \infty \end{cases} \tag{23}$$

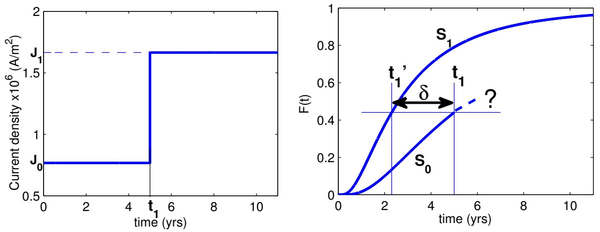
where $J_0 \neq J_1$ and t_1 is large, such that many conductors may have failed before t_1. This current profile is shown in Fig. 2a. The population S_0 is *fresh* at $t = 0$, but as time progresses, it suffers damage due to EM and the conductors start failing. We are interested in determining the distribution of the RV that describes the statistics of TTFs of the population.

Unfortunately, the *effective-EM* current model (5) is not applicable to this case because it implicitly assumes that the resulting *effective-EM* current density is applied to all the conductors throughout their lifetime. It is meant to handle current waveforms that change on a much smaller time-scale as compared to the lifetime of the conductors; whereas in our experiment the current changes on a time-scale that is comparable to the TTF of the conductors. Hence, many conductors might have failed exclusively due to J_0. This motivates the need for a new approach to estimate the statistics of the surviving sub-population.

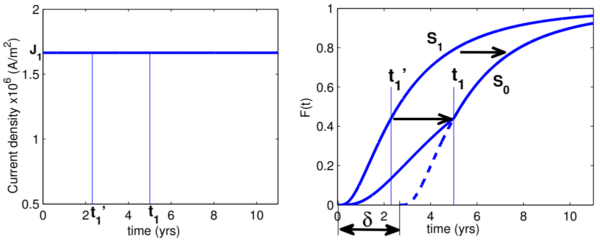
To motivate such an approach, consider another set S_1 of N conductors identical to those in S_0. Suppose S_1 is subjected to a current density of J_1 for all $t \geq 0$, as shown in Fig. 2b. Let $F_1(t)$ be the cumulative distribution function (cdf) of the population S_1 and $F_0(t)$ be the cdf of S_0. Clearly, $F_1(t)$ is known to be a lognormal. Define t_1' to be such that $F_1(t_1') = F_0(t_1)$ as shown in Fig. 2c. The difference $\delta = t_1 - t_1'$ is easy to compute, as we will demonstrate later. For now, we focus on the key question: 'what is the failure distribution of S_0 after t_1?' As already pointed out, traditional EM work is *not* helpful here. We now provide a proposal for answering this question.

Considering the two populations a) S_0 at time t_1, and b) S_1 at time t_1', notice that:

1. The two populations started out fresh with the same number of conductors, and the expected number of surviving members of the two populations are exactly *the same*, due to the fact that $F_1(t_1') = F_0(t_1)$. Therefore, the two populations have experienced an identical level of deterioration.



(a) Current density profile for S_0 (c) what is the cdf for S_0 for $t>t_1$?



(b) Current density profile for S_1 (d) The proposed solution

Figure 2: Proposed approach for single-step case, cdf $\approx$ no. of failed conductors/N

2. The two populations are subjected to exactly *the same* current stress J_1, as they move forward in time, i.e. $t_1 + x$ and $t_1' + x$, with $x \geq 0$ for S_0 and S_1, respectively.

Therefore, we expect that, going forward in time, both populations will see the same instantaneous failure rate, i.e.:

$$\begin{aligned} \lambda_0(t_1 + x) &= \lambda_1(t_1' + x), \quad \forall x \geq 0 \\ or \quad \lambda_0(t) &= \lambda_1(t - \delta) \quad \text{for } t > t_1 \end{aligned} \tag{24}$$

Since $\lambda_1(t)$ is the failure rate of a lognormal distribution, it follows that the failure rate of the surviving sub-population of S_0, i.e., $\lambda_0(t_1 + x)$, is that of a lognormal. Thus, we propose that the statistics of the surviving sub-population of S_0 be obtained by shifting the origin of the lognormal that gives rise to $\lambda_1(t)$ by δ so that the continuity of $F_0(t)$ at $t = t_1$ is maintained, as shown in Fig. 2d. The cdf $F_0(t)$ *needs to be continuous* because the TTF of a conductor can assume any value in a given time-range. Hence, for $t > t_1$, the statistics of S_0 are described by a section of a shifted lognormal distribution, the mean of which is identical to the mean of the lognormal that gave rise to $\lambda_1(t)$. This key point motivates the first two assumptions to be made in the next section.

We define two RVs $\mathbf{T}_0$ and $\mathbf{T}_1$, where $\mathbf{T}_0$ describes the TTF distribution of S_0 when it is subjected to J_0 for $t \geq 0$, and $\mathbf{T}_1$ describes the TTF distribution when S_0 is subjected to zero current density for $t \leq \delta$ and J_1 for $t > \delta$. The cdfs $F_{T_0}(t)$ and $F_{T_1}(t)$ of $\mathbf{T}_0$ and $\mathbf{T}_1$ are known to be lognormal. Following the arguments of the previous paragraph, we propose the cdf of S_0 to be:

$$F_0(t) = \begin{cases} F_{T_0}(t), & 0 \leq t \leq t_1 \\ F_{T_1}(t - \delta), & t_1 < t < \infty \end{cases} \tag{25}$$

where the time-shift $\delta \in (-\infty, t_1)$ is found using the *continuity constraint* $F_{T_1}(t_1 - \delta) = F_{T_0}(t_1)$ or:

$$\Phi\left[\frac{\ln(t_1 - \delta) - \mu_{\text{ln},1}}{\sigma_{\text{ln},1}\sqrt{2}}\right] = \Phi\left[\frac{\ln t_1 - \mu_{\text{ln},0}}{\sigma_{\text{ln},0}\sqrt{2}}\right] \tag{26}$$

Φ the is standard normal cdf, $\mu_{\text{ln,k}} = E[\ln \mathbf{T}_k]$ and $\sigma^2_{\text{ln,k}} = \text{Var}(\ln \mathbf{T}_k)$. Also, define $\mu_{\text{T,k}} = E[\mathbf{T}_k]$. Since Φ is monotonic, we equate the terms in the brackets and use (3) in (26) to arrive at:

$$\ln\left(\frac{t_1 - \delta}{\mu_{\text{T},1}}\right) = \frac{\sigma_{\text{ln},1}}{\sigma_{\text{ln},0}} \ln\left(\frac{t_1}{\mu_{\text{T},0}}\right) + 0.5\sigma_{\text{ln},1}(\sigma_{\text{ln},0} - \sigma_{\text{ln},1}) \tag{27}$$

Since $\mathbf{T}_0$ and $\mathbf{T}_1$ describe the statistics for the same set S_0, $\sigma_{\text{ln},0} = \sigma_{\text{ln},1}$. Also, we know that $(\mu_{\text{T},1}/\mu_{\text{T},0}) = (J_0/J_1)^n$, and hence the relative time-shift δ between $\mathbf{T}_1$ and $\mathbf{T}_0$ is:

$$\delta = t_1 \left[1 - \left(\frac{J_0}{J_1}\right)^n\right] \tag{28}$$

We next summarize the assumptions used to arrive at the cdf of S_0 and use them rigorously to obtain the cdf in the general case of multiple change in currents.

3.3.2 Assumptions

Consider a population set S that has already been subjected to some prior current density stress, i.e. it is not *fresh*. We make the following mild assumptions about S for some time $t \neq 0$:

ASSUMPTION 1. *The statistics of the TTFs for the surviving population of S are described by a (section of) a shifted lognormal distribution.*

ASSUMPTION 2. *The mean of the shifted lognormal distribution (relative to its start time) is given by Black's equation, with J being the current density at time t.*

ASSUMPTION 3. *The value of σ_{ln} for S at time t is the same as that of the fresh population at $t = 0$.*

We note that assumption 3 can be dropped if the dependence of σ_{ln} with regard to the damage accumulated due to EM is empirically known beforehand.

3.3.3 The case of multiple change in currents

Consider a second thought experiment with S_0 in which the current density profile is given as:

$$J(t) = J_k, \quad t_k < t \leq t_{k+1}, \quad k = 0, 1 \ldots n \tag{29}$$

where $J_{k-1} \neq J_k \; \forall k > 0$, $t_0 = 0$ and $t_{n+1} = \infty$. It is interesting to note that (29) is the typical current density profile of a surviving interconnect in the power grid, where the k^{th} failing interconnect has $\tau = t_k$.

As per Assumption 1, for each time-span $t_k < t \leq t_{k+1}$, the statistics are described by a RV $\mathbf{T}_k$ which has a lognormal distribution originating at some $t = \Delta_k$. Assumption 2 dictates that the mean $\mu_{\mathrm{T,k}}$ is given by Black's Eq. with $J = J_k$. In order to satisfy the *continuity constraint*, each RV $\mathbf{T}_k$ has a time-shift of δ_k with respect to $\mathbf{T}_{k-1}$, with $\mathbf{T}_0$ having a shift of $\delta_0 = 0$. This implies that the distribution for $\mathbf{T}_k$ originates at $\Delta_k = \sum_{i=0}^{k} \delta_i$. The cdf of S_0 can now be written as:

$$F_0(t) = F_{T_k}(t - \Delta_k), \quad t_k < t \leq t_{k+1}, \quad k = 0, 1 \ldots n \tag{30}$$

The time-shift δ_k between $\mathbf{T}_k$ and $\mathbf{T}_{k-1}$ can be found using the *continuity constraint*:

$$\Phi\left[\frac{\ln(t_k - \Delta_k) - \mu_{\mathrm{ln,k}}}{\sigma_{\mathrm{ln,k}}\sqrt{2}}\right] = \Phi\left[\frac{\ln(t_k - \Delta_{k-1}) - \mu_{\mathrm{ln,k\text{-}1}}}{\sigma_{\mathrm{ln,k\text{-}1}}\sqrt{2}}\right]$$

Using $\Delta_k = \Delta_{k-1} + \delta_k$, assumption 3 ($\sigma_{\mathrm{ln,0}} = \ldots = \sigma_{\mathrm{ln,n}} = \sigma_{\mathrm{ln}}$) and (3), we have:

$$\ln\left(\frac{t_k - \Delta_{k-1} - \delta_k}{\mu_{\mathrm{T,k}}}\right) = \ln\left(\frac{t_k - \Delta_{k-1}}{\mu_{\mathrm{T,k\text{-}1}}}\right)$$

Equating the terms in brackets, we have:

$$\delta_k = (t_k - \Delta_{k-1})(1 - r_k) = \left(t_k - \sum_{i=1}^{k-1} \delta_i\right)(1 - r_k) \tag{31}$$

where $\left(\frac{\mu_{\mathrm{T,k}}}{\mu_{\mathrm{T,k\text{-}1}}}\right) = \left(\frac{J_{k-1}}{J_k}\right)^n = r_k$.

3.3.4 Incorporating Blech Effect

The previous analysis assumes $J_k L > \beta_c \; \forall k$. However, due to change in current density, we might have *EM-immune* and *EM-susceptible* time-spans interspersed with each other. Let M and B be the set of integers k, where k denotes the time-span $t_k < t \leq t_{k+1}$, so that $M = \{k : J_k L \leq \beta_c\}$ and $B = \{k : J_k l > \beta_c\}$. Clearly, $M \cap B = \emptyset$ and $M \cup B$ is the entire time period.

We extend the framework developed so far to incorporate the *Blech effect* by introducing two modifications. First, assumption 1 is now applicable only for *EM-susceptible* time-spans, so that RV $\mathbf{T}_k$, that has a shifted lognormal distribution originating at $t = \Delta_k$, exists for $k \in B$. Since the current densities in the *EM-immune* time-spans cannot generate sufficient stress for a surviving conductor to fail, the corresponding probability of failure is zero and the associated cdf is a constant function. As we move forward in time, the conductors start failing again when $k \in B$ is encountered. Thus, the cdf of S_0 in this case is found to be:

$$F_0(t) = \begin{cases} 0 & k \in M : k < \min(B) \\ F_{T_b}(t_{b+1} - \Delta_b), & k \in M, b \in B \\ F_{T_k}(t - \Delta_k), & k \in B \end{cases} \tag{32}$$

where $b \in B : b < k$ and $|k - b|$ is minimum.

Second, the distribution for $\mathbf{T}_k$ now originates at

$$\Delta_k = \Theta(t_k) + \textstyle\sum_{i=0, i \in B}^{k} \delta_i \tag{33}$$

where $\Theta(t) = \sum_{k \in Q}(t_{k+1} - t_k)$ s.t. $Q = \{k : k \in M$ and $t_{k+1} < t\}$ and δ_k is defined as follows: consider a general scenario in which $(p - 1)$ consecutive EM-immune time-spans are sandwiched between two EM-susceptible time-spans. To be precise, $\{k - p, k\} \in B$ and $\{k - p + 1, \ldots k - 1\} \in M$. Then, δ_k is time-shift of $\mathbf{T}_k$ relative to $\mathbf{T}_{k-p}$ needed to maintain the *continuity constraint* if the in-between *EM-immune* time-spans were removed and is found to be:

$$\delta_k = (t_{k-p+1} - \Delta_{k-p})(1 - (J_{k-p}/J_k)^n) \tag{34}$$

Note that (34) reduces to (31) for $p = 1$.

3.3.5 Updating a TTF sample

Consider a conductor $\mathcal{C}$ of the set S_0 subjected to the current density profile (29). Clearly, the TTF of $\mathcal{C}$ changes because the RV describing the statistics of the population changes for each time-span. Assume, for the sake of argument, that $\mathcal{C}$ survives for $t > t_k$ and $0, k \in B$. At $t_0 (= 0)$, $\mathcal{C}$ has a TTF given by (using (4)):

$$\tau_0 = \mu_{\mathrm{T,0}}.\exp\left(\Psi\sigma_{\mathrm{ln}} - 0.5\sigma_{\mathrm{ln}}^2\right) \tag{35}$$

where the symbols are as defined before. At $t = t_k$, when the k^{th} current change occurs, the TTF of $\mathcal{C}$ is updated using the following relation:

$$\tau_k = \Delta_k + \mu_{\mathrm{T,k}} \exp\left(\Psi\sigma_{\mathrm{ln}} - 0.5\sigma_{\mathrm{ln}}^2\right) \tag{36}$$

where Ψ is the same sample value from Φ as used in (35). The offset Δ_k is added so that τ_k is now referred from $t = 0$. For $k \in M$, τ_k is defined to be ∞.

THEOREM 1. *Consider a conductor having the current density profile of (29). Let $\{k - p, k\} \in B$ and $\{k - p + 1, \ldots k - 1\} \in M$. Then, if eq. (34) and (36) are used to find the offset (δ_k) and TTF (τ_k) for the conductor, we always have:*

$$\tau_k = t_k + (\tau_{k-p} - t_{k-p+1})\left(\frac{J_{k-p}}{J_k}\right)^n$$

so that $\tau_k > t_k$.

The proof of this result is given in the Appendix.

4. IMPLEMENTATION

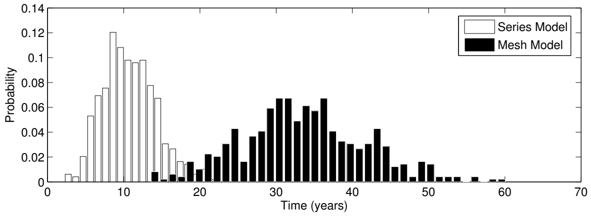
The overall flow for obtaining a sample of power grid TTF using the *series* and *mesh* model is given in Algorithm 1. We start by assigning TTF samples to all the resistors in the power grid using a Random Number Generator. Considering the grid as a series system, failure of the first resistor should cause the grid to fail. Hence τ_s is assigned the TTF sample of the first resistor in the sorted list R_{list}. In order to find the TTF sample of the grid using *mesh* model (τ_m), we start failing resistors. The voltage drops are efficiently updated using the *Woodbury-Banachiewicz-Schur* formulation (Algorithm 2). We also update the TTFs of surviving interconnects as outlined in section 3.3 and sort the surviving resistors as per their updated TTFs to determine the next resistor to fail. We keep failing resistors until either node

Algorithm 1 TTF_SAMPLE_GRID

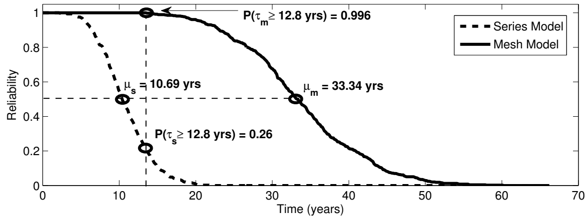
```
Input: $L_G, U_G, \mathrm{R}_{list}, v_0, v_{th}, \beta_c$
Output: $\tau_m, \tau_s$ (Sample TTF using mesh and series model)
Use Random Number Generator to assign TTF samples to all resistors in the list $\mathrm{R}_{list}$ based on current desities at $t = 0$
Sort $\mathrm{R}_{list}$ in ascending order of TTF samples
$\tau_s \leftarrow \mathrm{R}_{list}[1]$.ttf
for $k = 1 \rightarrow \mathrm{SIZE}(\mathrm{R}_{list})$ do
    grid_failed $\leftarrow 0$
    Find $\Delta G_k$, the conductance stamp of $\mathrm{R}_{list}[k]$
    $\{v_k,$ grid_singular$\} \leftarrow \mathrm{WB}(v_0, \Delta G_k, L_G, U_G, Z, H, W_{inv}, y, \gamma, k)$
    if grid_singular $= 1$ then
        $\tau_m \leftarrow \mathrm{R}_{list}[k]$.ttf
        break the outer for loop
    end if
    for $q = 1 \rightarrow \mathrm{SIZE}(v_k)$ do
        if $v_k[q] \geq v_{th}[q]$ then
            $\tau_m \leftarrow \mathrm{R}_{list}[k]$.ttf
            grid_failed $\leftarrow 1$
            break the inner for loop
        end if
    end for
    if grid_failed $= 1$ then
        break the outer for loop
    end if
    for $q = k + 1 \rightarrow \mathrm{SIZE}(\mathrm{R}_{list})$ do
        Update the TTF of $\mathrm{R}_{list}[q]$ as outlined in section 3.3.
    end for
    Sort $\mathrm{R}_{list}$ from $k + 1 \rightarrow \mathrm{SIZE}(\mathrm{R}_{list})$ in ascending order of TTF samples
end for
```

Algorithm 2 WB (Woodbury-Banachiewicz-Schur Voltage Update)

```
Input: $v_0, \Delta G, L_G, U_G, Z, H, W_{inv}, y, \gamma, k$
Output: $v_k$, grid_singular
Find $u$ and $h$ s.t. $\Delta G = uh^T$ as shown in (8)
$z \leftarrow$ BF_SUBSTITUTION$(L_G, U_G, u)$
Append $z$ as the $k^{th}$ column of $Z$
Append $h$ as the $k^{th}$ column of $H$
$p \leftarrow -u^T v_0$
Update $y$ by appending $p$ as the $k^{th}$ element as given in (19)
if $k = 1$ then
    $W_{inv} \leftarrow 1/\left(1 - u^T z\right)$
    $\gamma \leftarrow W_{inv} y$
else
    Find $b$ and $d$ as given in equation (16)
    $\mathcal{W}_b \leftarrow W_{inv} b$
    $s \leftarrow d - b^T \mathcal{W}_b$
    if $s = 0$ then
        grid_singular $\leftarrow 1$
        return
    end if
    $a \leftarrow \dfrac{b^T \gamma - p}{s}$
    $\gamma \leftarrow \gamma + a.\mathcal{W}_b$
    Append $-a$ as the $k^{th}$ element of $\gamma$ as shown in (22)
    Update $W_{inv}$ as given in (17)
end if
$v_k \leftarrow v_0 - Z.\gamma$
```



(a) Probability Distribution Function for TTFs of External Grid IBMPG5



(b) Reliability Function of External Grid IBMPG5

Figure 3: Estimated statistics for IBMPG5 ($\sim 250k$ nodes)

voltage drop(s) exceeds v_{th} or the grid becomes *singular*, after which the grid is deemed to have failed. τ_m is assigned the TTF sample of the last resistor, that caused the grid to fail.

We estimate the MTF (μ) of a grid by *random sampling*. In other words, Algorithm 1 is run w times to generate w grid TTF samples. The arithmetic mean of these samples is the estimate of μ. Suppose we were sampling from a *normal* distribution, whose variance is unknown. Then, in order to ensure an upper bound ϵ on relative error between arithmetic mean $\bar{x}_w$ and the true mean μ with a confidence of $(1 - \alpha) \times 100\%$, the number of samples w needed is given by [13]:

$$w \geq \left(\frac{z_{\alpha/2} s_w}{|\bar{x}_w| \epsilon/(1 + \epsilon)}\right)^2 \tag{37}$$

where s_w^2 is the unbiased estimator of variance and $z_{\alpha/2}$ is the $(1 - \alpha/2)$-percentile of Φ. The *central limit theorem* states that for large w, all distributions approach the normal. In practice, for w as large as 25 or 30 (conservatively we use $w \geq 50$), the t-distribution given by $[\bar{x}_w - \mu]/\left(s_w \sqrt{w}\right)$ is fairly close to a normal. Hence, we use (37) to estimate the mean for both *series* model (μ_s) and *mesh* model (μ_m), with $\alpha = 0.05$ and $\epsilon = 0.05$.

We also use *Monte Carlo random sampling* to estimate the probability of survival of a given grid upto a period of $\mathcal{Y}$ years. Again, we generate w grid TTF samples using Algorithm 1. For each sample, we have a *success* if the grid survives upto $\mathcal{Y}$ years, else we have a *failure*. By the law of large numbers, the fraction $\mathcal{P}$ = no. of successes/w, will converge to the true survival probability as $w \rightarrow \infty$. The stopping criteria (i.e. number of samples w required) is defined such that we have $(1 - \alpha) \times 100\%$ confidence that the error in probability estimation is less than $\mathcal{E}$ [14]:

$$w = MAX\left[\left(z_{\alpha/2}/(2\mathcal{E})\right)^2, \left((\sqrt{63} + z_{\alpha/2})/(2\sqrt{\mathcal{E}})\right)^2, \left(\left(z_{\alpha/2}\sqrt{2\mathcal{E} + 0.1} + \sqrt{(\mathcal{E} + 0.1) z_{\alpha/2}^2 + 3\mathcal{E}}\right)/(2\mathcal{E})\right)^2\right] \tag{38}$$

By choosing $\alpha = 0.05$ and $\mathcal{E} = 0.05$, we get $w = 489$ for all grids. *Reliability* of a Power Grid is essentially it's *Survival Probability* at different points in time. Hence, if the statistics of TTFs of the power grid following the *mesh* and *series* model is described by the RVs $\mathbf{X}_m$ and $\mathbf{X}_s$, then their *Probability Distribution Function* and *Reliability* can be obtained by calculating the survival probabilities for different values of $\mathcal{Y}$.

5. EXPERIMENTAL RESULTS

A `C++` implementation was written to test the proposed approach. Two types of test grids were used. The first type were *generated* per user specifications, including grid dimensions, metal layers, pitch and width per layer. The supply voltages and current sources were randomly placed on the grid. The technology specifications were consistent with 1.1 V 65nm CMOS technology. These grids are henceforth referred to as *internal* grids. The second type of grids are part of IBM power grid benchmarks [16]. These grids are *dual grids*, but the proposed approach was tested only for v_{dd} part of the grids and are referred to as *external* grids. The *Voltage drop threshold* was defined to be 10% of v_{dd} for *all* nodes in a grid. For concreteness in results, we assume the

Table 1: Comparison of Power Grid MTF as estimated using the *series* model and *mesh* model

Power Grid		max(v_0)		series model		mesh model				Gain	Net CPU
Name	Nodes[a]	volts	$\%v_{dd}$	μ_s (yrs)	TTF Samples[b]	μ_m (yrs)	TTF Samples[b]	Avg fails[c]	Time/iter -ation	Ratio (μ_m/μ_s)	Time
IBMPG2	61,677	0.125	6.97	10.81	165	35.67	53	19.32	5.80 sec.	3.30	5.18 min.
IBMPG3	410,011	0.122	6.78	10.37	98	46.84	73	50.29	2.56 min.	4.52	3.12 hr.
IBMPG4	474,524	0.132	7.35	9.58	160	36.27	82	40.55	2.60 min.	3.79	3.55 hr.
IBMPG5	248,838	0.086	4.80	10.69	169	33.34	102	17	23.46 sec.	3.12	39.97 min.
IBMPG6	403,915	0.103	5.73	11.20	195	44.28	79	30.89	1.58 min.	3.95	2.08 hr.
IBMPGNEW1	315,951	0.168	9.36	12.94	172	59.96	92	60.80	2.80 min.	4.64	4.29 hr.
IBMPGNEW2	717,754	0.139	7.72	10.12	164	39.64	75	41.97	3.74 min.	3.92	4.68 hr.
G1	50,444	0.040	3.64	10.94	171	34.46	105	19.24	1.92 sec.	3.15	3.46 min.
G2	113,304	0.050	4.56	10.98	131	35.07	101	27.28	15 sec.	3.19	25.32 min.
G3	200,828	0.055	4.98	9.09	140	29.35	91	43.75	45.50 sec.	3.23	1.15 hr.
G4	449,182	0.050	4.58	10.85	127	44.57	68	85.16	3.42 min.	4.11	3.88 hr.
G5	1,006,625	0.057	5.18	12.45	101	46.90	97	85.56	8.34 min.	3.77	13.48 hr.

[a] Number of nodes in v_{dd} rails after merging the short paths as equivalent node

[b] Number of grid TTF samples required to estimate the MTF with 95% confidence and max 5% relative error

[c] Average number of interconnect failures before the grid fails in each Monte Carlo pass for *mesh* model

Table 2: Survival Probability Estimation

Power Grid	max(v_0)		$\mathcal{Y}$	$\mathcal{P}_s$	$\mathcal{P}_m$	CPU
	volts	$\%v_{dd}$	(yrs)	$\mathcal{P}(\tau_s \geq \mathcal{Y})$	$\mathcal{P}(\tau_m \geq \mathcal{Y})$	Time
IBMPG2	0.125	6.97	11.61	0.413	1.000	4.86 min
IBMPG3	0.122	6.77	12.50	0.256	1.000	64.73 min
IBMPG4	0.132	7.35	12.10	0.178	1.000	1.61 hr.
IBMPG5	0.086	4.80	12.80	0.260	0.996	28.95 min
IBMPG6	0.103	5.73	14.00	0.200	1.000	1.15hr.
IBMPGNEW1	0.168	9.36	18.50	0.088	1.000	1.30 hr.
IBMPGNEW2	0.139	7.72	15.00	0.049	1.000	3.02 hr.

following configuration: the interconnect material is Aluminum (Al), with activation energy $E_a = 0.9eV$, $\beta_c = 3000A/cm$ and a current exponent $n = 1$. A nominal interconnect temperature of $373K$ was used in simulations. The standard deviation of the lognormal $\sigma_{\ln}$ was assumed to be 0.81 for all interconnects in the grid, consistent with typical data in the literature. We used a 2.6 GHz Linux machine with 24 GB of RAM.

Table 1 compares the power grid MTF as estimated using the *series* model and *mesh* model, using a *gain ratio* (μ_m/μ_s). The gain ratio is dependent on $\Delta v = |v_{th} - v_0|$; as this difference increases, the gain ratio also increases. If the difference Δv is small, then the *mesh* model degenerates to *series* model. Given a reasonable difference between v_{th} and v_0 , the gain ratio is 3-4 for all the grids. Table 2 compares the survival probability of *external* grids based on user-specified values for $\mathcal{Y}$. It is seen that by taking redundancies into account, the *mesh* model consistently predicts a higher survival probability as compared to *series*. For a given grid, the time required to estimate the survival probability using *mesh* model increases with increase in $\mathcal{Y}$, but it also enables us to estimate the reliability of the grid $\forall t \leq \mathcal{Y}$. For a complete overview, Fig. 3 plots the probability distribution function (pdf) and reliability of IBMPG5 grid as estimated using the *series* and *mesh* model. Clearly, the *series* model gives a pessimistic estimate of power grid TTF statistics. Also, using *goodness-of-fit* methods, it was found that the RV $\mathbf{X}_m$ has a *normal* distribution, thus justifying our use of (37) to estimate MTF.

Each *Monte Carlo* iteration gives us a grid TTF sample. Since *Monte Carlo* estimation approach is highly parallelizable (i.e. the samples can be obtained concurrently), we determine the scalability of the proposed solution by plotting the time taken per iteration (as given in column *Time/iteration* of Table 1) vs. number of nodes. As shown in figure 4, the run-time is found to be roughly $O(m^{1.4})$, where m is the size of grid in number of nodes.

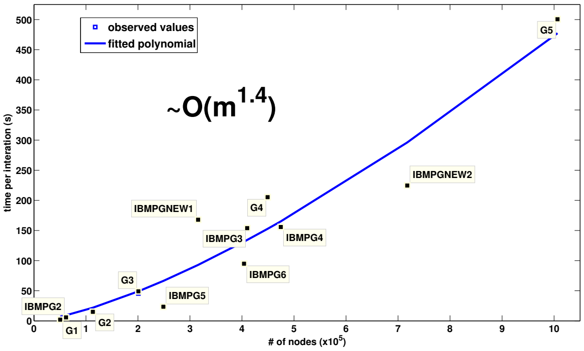


Figure 4: Scalability of proposed solution

Finally, other than Δv, there are many parameters which can potentially affect the *gain ratio*. From (1) and (35), we observe that Black's constant A, Activation energy E_a and Temperature of metal T_m only scale the TTFs of all interconnects *linearly*, hence a change in values of A, E_a or T_m will not alter the sequence of failure of interconnects. As such, the *gain ratio* will be unchanged. However, due to non-linear dependence, the sequence may change with change in standard deviation of the lognormal (σ_{ln}), current exponent (n) and resistivity of interconnects (ρ). Figure 5 shows the effect of varying these parameters and β_c on the gain ratio of grid G1. In each case, *mesh* model predicts a more reliable grid than the *series* model and thus supports our claim that a lot of EM-margin is left 'on the table'.

6. CONCLUSIONS

We proposed and implemented 1) the *mesh* model to estimate the MTF and reliability of power grid, supplemented by a fast and *exact* approach to update the voltage drops using Woodbury formula and Banachiewicz-Schur form and 2) a framework to predict the failure statistics of a metal line as it's *effective-EM* current density varies. The *mesh* model accounts for the redundancies in the grid based on the new failure criteria of node *voltage drop threshold*, and thus obtains a more realistic estimate of grid's MTF and reliability. The results prove that the current practice of ignoring the redundancy in the grid gives a pessimistic estimate of grid TTF statistics by a factor of 3-4.

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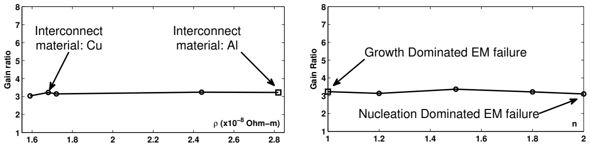
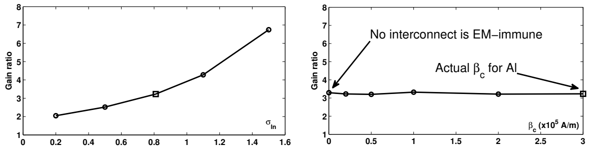


(a) Gain ratio vs. resistivity (ρ) of interconnects

(c) Gain ratio vs. current−exponent (n)



(b) Gain ratio vs. std dev of lognormal (σ$_{ln}$)

(d) Gain ratio vs. Critical Blech product (β$_c$)

Figure 5: Estimated gain ratio of G1 under different parameter values; the square markers represent the parameters used for Table 1

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APPENDIX

A. THE BANACHIEWICZ-SCHUR FORM

Let $M \in \mathbb{R}^{k\times k}$ be 2×2 block matrix:

$$M = \begin{bmatrix} A & b \\ c^T & d \end{bmatrix} \tag{39}$$

where $A \in \mathbb{R}^{(k-1)\times(k-1)}$, $b \in \mathbb{R}^{k-1}$, $c \in \mathbb{R}^{k-1}$, and d is a scalar. The Schur-complement of A in M is given by [17]:

$$s = d - c^T A^{-1} b \tag{40}$$

If both M and A in (39) are non-singular, then s is non-singular too, i.e. $s \neq 0$. This allows writing M as:

$$M = \begin{bmatrix} I_{k-1} & 0 \\ c^T A^{-1} & I_1 \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & s \end{bmatrix} \begin{bmatrix} I_{k-1} & A^{-1}b \\ 0 & I_1 \end{bmatrix} \tag{41}$$

where I_λ is the identity matrix of order λ. The inverse of M can now be written as [17]:

$$M^{-1} = \begin{bmatrix} I_{k-1} & -A^{-1}b \\ 0 & I_1 \end{bmatrix} \begin{bmatrix} A^{-1} & 0 \\ 0 & 1/s \end{bmatrix} \begin{bmatrix} I_{k-1} & 0 \\ -c^T A^{-1} & I_1 \end{bmatrix} \tag{42}$$

which can be re-written as:

$$M^{-1} = \begin{bmatrix} A^{-1} + \dfrac{A^{-1}bc^TA^{-1}}{s} & -\dfrac{A^{-1}b^T}{s} \\ -\dfrac{c^TA^{-1}}{s} & \dfrac{1}{s} \end{bmatrix} \tag{43}$$

Equation (43) is known as the *Banachiewicz-Schur form.* It expresses M^{-1} in terms of A^{-1}.

B. PROOF OF THEOREM 1

Let $\Upsilon = exp\left(\Psi\sigma_{\text{ln}} - 0.5\sigma_{\text{ln}}^2\right)$. By Assumption 3, σ_{ln} is constant. Thus, Υ will be the same throughout the life-time of the conductor.

The conductor was EM-susceptible for $t_{k-p} < t \leq t_{k-p+1}$, became *EM-immune* for all time span(s) following $t = t_{k-p+1}$ until $t = t_k$, when it becomes *EM-susceptible* again. Clearly, $\tau_{k-p} > t_{k-p+1}$, or the conductor should have failed before t_{k-p+1}. From theory, the RVs $\mathbf{T}_{k-p+1}, \ldots \mathbf{T}_{k-1}$ are undefined. From (33), the time of origin of $\mathbf{T}_k$ can be written as:

$$\begin{aligned} \Delta_k &= \Delta_{k-p} + \Theta(t_k) - \Theta(t_{k-p}) + \delta_k \\ &= \Delta_{k-p} + (t_k - t_{k-p+1}) + (t_{k-p+1} - \Delta_{k-p})(1 - r_k) \\ &= t_k - r_k t_{k-p+1} + r_k \Delta_{k-p} \qquad \text{(i)} \end{aligned}$$

where $r_k = \left(\dfrac{J_{k-p}}{J_k}\right)^n = \left(\dfrac{\mu_{\text{T,k}}}{\mu_{\text{T,k-p}}}\right)$. The new TTF at $t = t_k$ can now be written as:

$$\begin{aligned} \tau_k &= \Delta_k + \mu_{\text{T,k}}\Upsilon \\ &= t_k - r_k t_{k-p+1} + r_k\Delta_{k-p} + r_k \mu_{\text{T,k-p}}\Upsilon \qquad \text{(from (i))} \\ &= t_k + (\Delta_{k-p} + \mu_{\text{T,k-p}}\Upsilon - t_{k-p+1})r_k \\ &= t_k + (\tau_{k-p} - t_{k-p+1})r_k \qquad \text{(using (36))} \end{aligned}$$

$\square$