ECE 454
Computer Systems Programming

Optimizing for Caches

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Content

• Cache basics and organization (last lecture)
• Optimizing for caches (this lecture)
  • Loop reordering
  • Tiling/blocking
• Prefetching (later)
• Virtual memory (later)
Optimizing for Caches
Memory Optimizations

• **Write code that has locality**
  • Spatial: access data that is contiguous as much as possible
  • Temporal: access the same data within short intervals of time

• **How to achieve locality?**
  • Proper choice of algorithm
  • Loop transformations
Background: Array Allocation

- \( T \ A[L] \);
  - Array of data type \( T \) and length \( L \)
  - Contiguously allocated region of \( L \times \text{sizeof}(T) \) bytes

```
char string[12];

int val[5];

double a[3];

char *p[3];
(64 bit)
```
Multidimensional (Nested) Arrays

- Declaration: \( T \ A[R][C]; \)
  - 2D array of data type \( T \)
  - \( R \) rows, \( C \) columns
  - \( T \) element requires \( K \) bytes

- Arrangement
  - Row-Major Ordering (C code)

\[
\begin{align*}
A[0][0] & \quad \ldots \quad A[0][C-1] \\
\vdots & \quad \quad \quad \vdots \\
A[R-1][0] & \quad \ldots \quad A[R-1][C-1]
\end{align*}
\]

Array Size: \( R \times C \times K \) bytes
Assumed Simple Cache

- 2 (4 byte) integers per block (8 bytes)
- 2-way set associative (2 blocks per set)
- 1 set
- Total size = 2 blocks, i.e., same as fully associative
- Replacement policy: Least Recently Used (LRU)
Some Key Questions

• How many elements are there per block?
• Does the data structure fit in the cache?
• Do I reuse blocks over time?
• In what order am I accessing blocks?
Simple Array

Cache

<table>
<thead>
<tr>
<th>Block 0</th>
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<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Block 1</td>
<td></td>
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<td></td>
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</tr>
</tbody>
</table>

for (i=0; i<N; i++){
    ... = A[i];
}

- Miss rate = #misses / #accesses = \( \frac{N/2}{N} = \frac{1}{2} = 50\% \)
Simple Array with Outer Loop, Fits in Cache

- Assume array A fits in the cache
- Miss rate = \#misses / \#accesses = \( \frac{N/2}{NP} = \frac{1}{2P} \)

Lesson: for sequential accesses with re-use, if data fits in the cache, first visit suffers all the misses
Simple Array with Outer Loop, Doesn’t Fit in Cache

- Assume array A does not fit in the cache
- Miss rate = #misses / #accesses = \( \frac{N/2}{N} = \frac{1}{2} = 50\% \)

Lesson: for sequential accesses with re-use, if the data doesn’t fit, **same** miss rate as no-reuse
2D Array, Fits in Cache

- Assume matrix A fits in the cache
- Miss rate = #misses / #accesses = \( \frac{(N^2/2)}{N^2} = \frac{1}{2} = 50\% \)
2D Array, Doesn’t Fit in Cache

• Assume matrix A does not fit in the cache

• Miss rate = #misses / #accesses = \( \frac{N^2}{2} / N^2 = 1/2 = 50\% \)

Lesson: for 2D accesses, row-order, no re-use, whether data fits or not, same rate as sequential
2D Array, Column Order, Column Fits in Cache

- Assume matrix A fits in the cache
- Miss rate = \#misses / \#accesses = \(\frac{(N^2/2)}{N^2} = \frac{1}{2} = 50\%\)

Lesson: for 2D accesses, column-order, no re-use, when column data fits in cache, same rate as sequential

\[
\begin{array}{c|cccc}
\text{Block 0} & 1 & 2 & 3 & 4 \\
\hline
\text{Block 1} & 5 & 6 & 7 & 8 \\
\end{array}
\]

\[
\text{for (j=0; j<N/2; j++) }
\text{for (i=0; i<N; i++)}
\text{\{ }
\text{... = A[i][j]; }
\text{\}}
\]

14
2D Array, Column Order, Column Doesn’t Fit in Cache

- Assume matrix A does not fit in the cache
- Miss rate = \#misses / \#accesses = \( \frac{N^2}{N^2} = 100\% \)

Lesson: for 2D accesses, column-order, no re-use, when column data doesn’t fit in cache, 100% miss rate
Loop Reordering/Interchange

- Initial code accesses data in column order
  - 100% miss rate, when column doesn’t fit in cache
- Loop reordering allows accessing data in row order
  - 50% miss rate

```c
for (i=0; i<N; i++) {
    for (j=0; j<N; j++){
        A[j][i] = i * j;
    }
}

for (j=0; j<N; j++) {
    for (i=0; i<N; i++){
        A[j][i] = i * j;
    }
}
```
Matrix Multiplication

for (i=0; i<N; i++){
    for (j=0; j<N; j++){
        for (k=0; k<N; k++){
            ... = A[i][k] * B[k][j];
        }
    }
}
2 2D Arrays

- Assume matrix A and matrix B do not fit in cache
- Assume a column of B does not fit at same time as a row of A
- Miss rate = \#misses / \#accesses =

```java
for (i=0;i<N;i++){
    for (j=0;j<N;j++){
        for (k=0;k<N;k++){
            ... = A[i][k] * B[k][j];
        }
    }
}
```
2 2D Arrays

The inner most loop (i=j=0):

\[ A[0][0] \times B[0][0], \quad A[0][1] \times B[1][0], \quad A[0][2] \times B[2][0], \quad A[0][3] \times B[3][0] \]
2 2D Arrays

for (i=0; i<N; i++){
    for (j=0; j<N; j++){
        for (k=0; k<N; k++){
            ... = A[i][k] * B[k][j];
        }
    }
}

The inner most loop (i=j=0):
A[0][0] * B[0][0], A[0][1] * B[1][0],
A[0][2] * B[2][0], A[0][3] * B[3][0]

Cache Time

<table>
<thead>
<tr>
<th>A</th>
<th>1</th>
<th>2</th>
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<tr>
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<th>B</th>
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2 2D Arrays

The inner most loop (i=j=0):

\[ A[0][0] \times B[0][0], \; A[0][1] \times B[1][0], \; A[0][2] \times B[2][0], \; A[0][3] \times B[3][0] \]

for (i=0; i<N; i++) {
    for (j=0; j<N; j++) {
        for (k=0; k<N; k++) {
            \[ ... = A[i][k] \times B[k][j] \]
        }
    }
}
2 2D Arrays

for (i=0; i<N; i++) {
    for (j=0; j<N; j++) {
        for (k=0; k<N; k++) {
            ... = A[i][k] * B[k][j];
        }
    }
}

Next time: (i=0, j=1):

A[0][0] * B[0][1], A[0][1] * B[1][1],

<table>
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<th>Cache</th>
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2 2D Arrays

for (i=0;i<N;i++)
    for (j=0;j<N;j++)
        for (k=0;k<N;k++)
            ... = A[i][k] * B[k][j];

Next time: (i=0, j=1):  
A[0][0] * B[0][1], A[0][1] * B[1][1],
2 2D Arrays

for (i=0; i<N; i++){
    for (j=0; j<N; j++){
        for (k=0; k<N; k++){
            ...
        }
    }
}

Next time: (i=0, j=1):
- A[0][0] * B[0][1], A[0][1] * B[1][1],
for (i=0; i<N; i++){
    for (j=0; j<N; j++){
        for (k=0; k<N; k++){
            ... = A[i][k] * B[k][j];
        }
    }
}

Next time: (i=0, j=1):
A[0][0] * B[0][1], A[0][1] * B[1][1],
2 2D Arrays

Next time: (i=0, j=1):
\[ A[0][0] \times B[0][1], A[0][1] \times B[1][1], A[0][2] \times B[2][1], A[0][3] \times B[3][1] \]

for (i=0; i<N; i++){
  for (j=0; j<N; j++){
    for (k=0; k<N; k++){
      \[ A[i][k] \times B[k][j] \]
    }
  }
}

Cache

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Time

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2D Arrays

for (i=0; i<N; i++) {
    for (j=0; j<N; j++) {
        for (k=0; k<N; k++) {
            ... = A[i][k] * B[k][j];
        }
    }
}

Next time: (i=0, j=1):

A[0][0] * B[0][1], A[0][1] * B[1][1],
2 2D Arrays

- Assume matrix A and matrix B do not fit in cache
- Assume a column of B does not fit at same time as a row of A
- Miss rate = #misses / #accesses = 75%

```
for (i=0;i<N;i++){
    for (j=0;j<N;j++){
        for (k=0;k<N;k++){
            ... = A[i][k] * B[k][j];
        }
    }
}
```

Next time: (i=0, j=1):

- $A[0][0] * B[0][1]$,
- $A[0][1] * B[1][1]$,
- $A[0][2] * B[2][1]$,
- $A[0][3] * B[3][1]$
/ * Multiply N x N matrices a and b  */

```c
mmm(double a[][N], double b[][N], double c[][N])
{
    int i, j, k;
    for (i = 0; i < n; i++)
        for (j = 0; j < n; j++)
            for (k = 0; k < n; k++)
                c[i][j] += a[i][k]*b[k][j];
}
```
Cache Miss Analysis, First Iteration

- **Assume:**
  - Matrix elements are doubles
  - Cache block 64B = 8 doubles
  - Cache capacity $<< n$ (much smaller than $n$)
    - i.e., can’t even hold an entire row in the cache!

- **On first iteration:**
  - How many misses?
  - $n/8 + n = 9n/8$ misses

- **At the end of first iteration:**
  - in cache
Cache Miss Analysis, Second Iteration

• Assume:
  • Matrix elements are doubles
  • Cache block 64B = 8 doubles
  • Cache capacity << n (much smaller than n)
    • i.e., can’t even hold an entire row in the cache!

• On second iteration:
  • How many misses?
  • $n/8 + n = 9n/8$ misses

• Total misses for matrix multiplication:
  • $(9n/8) \times n^2 = (9/8)\times n^3$
Improving Cache Reuse

• Misses are expensive
  • L1 cache reference: 1-4 ns (L1 cache size: 32 KB)
  • Main memory reference: 100 ns (memory size: 4-256 GBs)

• Matrix multiplication has lots of data re-use
  • Key idea: Try to use entire cache block once it is loaded
  • Challenge: We need to work with both rows and columns

• Solution:
  • Operate in sub-squares of the matrices
    • One sub-square per matrix should fit in cache simultaneously
    • Heavily re-use the sub-squares before loading new ones
  • Called Tiling or Blocking (a sub-square is a tile)
Idea for Tiled Matrix Multiplication

- Perform tile based mini-multiplications
  - E.g., $TA_{00} \times TB_{00}$, $TA_{00} \times TB_{01}$
  - $C[0][0] = 1 \times 16 + 2 \times 15 + 3 \times 14 + 4 \times 13$
     - Perform as part of $TA_{00} \times TB_{00}$
     - Perform as part of $TA_{01} \times TB_{10}$
Tiled Matrix Multiplication

/* Multiply N x N matrices a and b */
void mmm(double a[][N], double b[][N], double c[][N], int T) {
    int i, j, k;
    for (i = 0; i < n; i+T)
        for (j = 0; j < n; j+T)
            for (k = 0; k < n; k+T)
                /* T x T mini matrix multiplications */
                    for (i1 = i; i1 < i+T; i1++)
                        for (j1 = j; j1 < j+T; j1++)
                            for (k1 = k; k1 < k+T; k1++)
                                c[i1][j1] += a[i1][k1]*b[k1][j1];
}

Tile size T x T
Visualization

- First calculate $TC_{00} = C[0][0] - C[T-1][T-1]$
Next calculate $TC_{01} = C[0][T] - C[T-1][2T-1]$
Why Does Tiling Work?

- \( C[0][0] = 1 \times 16 + 2 \times 15 + 3 \times 14 + 4 \times 13 \)
  - Perform as part of \( TA_{00} \times TB_{00} \)
  - Perform as part of \( TA_{01} \times TB_{10} \)

- Still have to access \( B[ ] \) column-wise
- But now \( B[ ] \)'s cache blocks don’t get replaced
Tiling Cache Miss Analysis

- Assume:
  - Cache block = 8 doubles
  - Cache capacity << n (much smaller than n)
  - With 3 arrays, need to fit 3 tiles in cache
    - Cache capacity > 3T^2

- Misses per tile-iteration:
  - T^2/8 misses for each tile
  - (2n/T) * (T^2/8) = nT/4

- Total misses:
  - Tiled: (nT/4) * (n/T)^2 = (1/(4T)) * n^3
  - Untiled: (9/8) * n^3