Homework Assignment 1

ECE 1762 Algorithms and Data Structures
Fall Semester, 2016

Due: 4PM October 3 (in ECE1762 box, SF basement near SFB560)

Unless otherwise stated, all page numbers are from 2008 edition of Cormen, Leiserson, Rivest and Stein (parentheses contain page numbers from the 2001 edition). **Unless otherwise stated, for each algorithm you design you should give a detailed description of the idea, proof of correctness, termination, analysis and proof of time and space complexity. If not, your answer will be incomplete and you will miss credit. You are allowed to refer to pages in the textbook.** When requested, do not give C code but explain your algorithm briefly with pseudocode!

This is a long homework ... start immedietaly.

1. [**15 points**]
   (a) Prove by induction that there are $n!$ bijective functions from $\{1..n\}$ to $\{1..n\}$.
   (b) Prove the identity:
   $$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k}$$
   by induction.
   (c) Prove the identity in the previous question by showing that the two sides of the equation count the same sets in different ways. What happens when $n = 0$? How is that related to the base case in the first question?

2. [**15 points**] Sort the following 12 functions from asymptotically smallest to asymptotically largest, indicating ties if there are any. You do not need to turn in proofs (in fact, please don't turn in proofs), but you should do them anyway just for practice.

1	n^2	$\lg n$	$\lg^* n$
$\lg^* 2^n$	$e^{(1+O(1/n))^2}$	$\lfloor \lg(n!) \rfloor$	$\lfloor \lg n \rfloor !$
$n^{\lg n}$	$(\lg n)^n$	$(\lg n)^{\lg n}$	$e + O(1/n)$

   To simplify notation, write $f(n) \ll g(n)$ to mean $f(n) = o(g(n))$ and $f(n) \equiv g(n)$ to mean $f(n) = \Theta(g(n))$. For example, the functions $n^2, n, \binom{n}{2}, n^3$ could be sorted either as $n \ll n^2 \equiv \binom{n}{2} \ll n^3$ or as $n \ll \binom{n}{2} \equiv n^2 \ll n^3$

3. [**15 points**] Solve the following recurrences. State tight asymptotic bounds for each function in the form $\Theta(f(n))$ for some recognizable function $f(n)$. You will need to turn in proofs for your bounds. Assume reasonable but nontrivial base cases if none are supplied.
   (a) $A(n) = 5A(n/3) + n \lg n$

(b) $B(n) = \min_{0<k<n} (B(k) + B(n-1) + 1)$

(c) $C(n) = C(n-1) + 1/n$

4. **[20 points]** The following program determines the maximum value in an unordered array $A[1 \ldots n]$ of distinct elements:

```
1. max = −∞
2. for i=1 to n do
3.     compare A[i] to max
4.     if A[i] > max then
5.         max = A[i]
```

(a) If a number x is randomly chosen from a set of n distinct numbers, what is the probability that x is the largest in that set?

(b) When line 5 of the program is executed, what is the relationship between $A[i]$ and $A[j]$ for $1 \leq j \leq i$?

(c) For each i in the range $1 \leq i \leq n$, what is the probability that line 5 is executed?

(d) Let $s_1, s_2, \ldots, s_n$ be n random variables, where s_i represents the number of times (0 or 1) that line 5 is executed during the i-th iteration of the **for**-loop. What is $E[s_i]$?

(e) Let $s = s_1 + s_2 + \ldots + s_n$ be the total number of times that line 5 is executed during some run of the program. Prove that $E[s] = \Theta(\log n)$.

5. **[15 points]** In this problem, we consider a decomposition of a tree that may be useful when using a divide-and-conquer approach to solve some problem on that tree.

(a) Show that for any binary tree with n vertices, there is an edge such that removing it partitions the tree into two subtrees each with at most $3n/4$ vertices.

(b) Show that there are arbitrarily large binary trees for which the best partition factor is $2/3$ (it is not sufficient to display a single tree of some size). That is, show that there exists a binary tree with n vertices (for arbitrarily large n) where no edge can be removed to partition the tree into two subtrees, each of which has less than $2n/3$ vertices.

(c) Show that given any binary tree with n vertices, by removing $O(\log n)$ edges, we can partition the tree into two *forests* (collections of trees) A and B with numbers of vertices $\lceil n/2 \rceil$ and $\lfloor n/2 \rfloor$.

6. **[10 points]** This problem asks you to simplify some recursively defined boolean formulas as much as possible. That is, find a simpler formula for α_n and β_n than the ones given, ideally in terms of p and q. In each case, prove that your answer is correct. Each proof can be just a few sentences long, but it must be a *proof* (that is, induction, contradiction, series of implications, etc).

(a) Suppose $\alpha_0 = p$, $\alpha_1 = q$, and $\alpha_n = (\alpha_{n-2} \wedge \alpha_{n-1})$ for all $n \geq 2$. Simplify α_n as much as possible. *[Hint: What is α_5?]*

(b) Suppose $\beta_0 = p$, $\beta_1 = q$, and $\beta_n = (\beta_{n-2} \Rightarrow \beta_{n-1})$ for all $n \geq 2$. Simplify β_n as much as possible. *[Hint: What is β_5?]*

7. **[10 points]** The hypercube graph Q_n is a graph formed from the vertices and edges of an n-dimensional hypercube: Q_1 is a line segment, Q_2 is a square, Q_3 is a normal cube, and so forth. In general, Q_n is constructed from Q_{n-1} by duplicating the graph and connecting all corresponding vertices with a new edge. Prove that Q_n has a Hamitonian cycle for all $n \geq 2$.