Chapter 2

Examples of Solved Problems

This section presents some typical problems that the student may encounter, and shows how such problems can be solved. In addition to the identities given in Section 2.5, these examples also use an identity known as *consensus*, defined below.

17*a*. $x \cdot y + y \cdot z + \overline{x} \cdot z = x \cdot y + \overline{x} \cdot z$ *Consensus*

17*b*. $(x + y) \cdot (y + z) \cdot (\overline{x} + z) = (x + y) \cdot (\overline{x} + z)$

Example 2.1

Problem: Determine if the following equation is valid

$$\overline{x}_1\overline{x}_3 + x_2x_3 + x_1\overline{x}_2 = \overline{x}_1x_2 + x_1x_3 + \overline{x}_2\overline{x}_3$$

Solution: The equation is valid if the expressions on the left- and right-hand sides represent the same function. To perform the comparison, we could construct a truth table for each side and see if the truth tables are the same. An algebraic approach is to derive a canonical sum-of-product form for each expression.

Using the fact that $x + \overline{x} = 1$ (Theorem 8*b*), we can manipulate the left-hand side as follows:

$$\begin{aligned}
\text{LHS} &= \overline{x}_1\overline{x}_3 + x_2x_3 + x_1\overline{x}_2 \\
&= \overline{x}_1(x_2 + \overline{x}_2)\overline{x}_3 + (x_1 + \overline{x}_1)x_2x_3 + x_1\overline{x}_2(x_3 + \overline{x}_3) \\
&= \overline{x}_1x_2\overline{x}_3 + \overline{x}_1\overline{x}_2\overline{x}_3 + x_1x_2x_3 + \overline{x}_1x_2x_3 + x_1\overline{x}_2x_3 + x_1\overline{x}_2\overline{x}_3
\end{aligned}$$

These product terms represent the minterms 2, 0, 7, 3, 5, and 4, respectively.

For the right-hand side we have

$$\begin{aligned}
\text{RHS} &= \overline{x}_1x_2 + x_1x_3 + \overline{x}_2\overline{x}_3 \\
&= \overline{x}_1x_2(x_3 + \overline{x}_3) + x_1(x_2 + \overline{x}_2)x_3 + (x_1 + \overline{x}_1)\overline{x}_2\overline{x}_3 \\
&= \overline{x}_1x_2x_3 + \overline{x}_1x_2\overline{x}_3 + x_1x_2x_3 + x_1\overline{x}_2x_3 + x_1\overline{x}_2\overline{x}_3 + \overline{x}_1\overline{x}_2\overline{x}_3
\end{aligned}$$

These product terms represent the minterms 3, 2, 7, 5, 4, and 0, respectively. Since both expressions specify the same minterms, they represent the same function; therefore, the equation is valid. Another way of representing this function is by $\sum m(0, 2, 3, 4, 5, 7)$.

Example 2.2

Problem: Design the minimum-cost product-of-sums expression for the function $f(x_1, x_2, x_3, x_4) = \sum m(0, 2, 4, 5, 6, 7, 8, 10, 12, 14, 15)$.

Solution: The function is defined in terms of its minterms. To find a POS expression we should start with the definition in terms of maxterms, which is $f = \Pi M(1, 3, 9, 11, 13)$. Thus,

$$\begin{aligned} f &= M_1 \cdot M_3 \cdot M_9 \cdot M_{11} \cdot M_{13} \\ &= (x_1 + x_2 + x_3 + \overline{x}_4)(x_1 + x_2 + \overline{x}_3 + \overline{x}_4)(\overline{x}_1 + x_2 + x_3 + \overline{x}_4)(\overline{x}_1 + x_2 + \overline{x}_3 + \overline{x}_4)(\overline{x}_1 + \overline{x}_2 + x_3 + \overline{x}_4) \end{aligned}$$

We can rewrite the product of the first two maxterms as

$$\begin{aligned} M_1 \cdot M_3 &= (x_1 + x_2 + \overline{x}_4 + x_3)(x_1 + x_2 + \overline{x}_4 + \overline{x}_3) && \text{using commutative property } 10b \\ &= x_1 + x_2 + \overline{x}_4 + x_3\overline{x}_3 && \text{using distributive property } 12b \\ &= x_1 + x_2 + \overline{x}_4 + 0 && \text{using theorem } 8a \\ &= x_1 + x_2 + \overline{x}_4 && \text{using theorem } 6b \end{aligned}$$

Similarly, $M_9 \cdot M_{11} = \overline{x}_1 + x_2 + \overline{x}_4$. Now, we can use M_{11} again, according to property $7a$, to derive $M_{11} \cdot M_{13} = \overline{x}_1 + x_3 + \overline{x}_4$. Hence

$$f = (x_1 + x_2 + \overline{x}_4)(\overline{x}_1 + x_2 + \overline{x}_4)(\overline{x}_1 + x_3 + \overline{x}_4)$$

Applying $12b$ again, we get the final answer

$$f = (x_2 + \overline{x}_4)(\overline{x}_1 + x_3 + \overline{x}_4)$$

Example 2.3

Problem: A circuit that controls a given digital system has three inputs: x_1, x_2, and x_3. It has to recognize three different conditions:

- Condition A is true if x_3 is true and either x_1 is true or x_2 is false
- Condition B is true if x_1 is true and either x_2 or x_3 is false
- Condition C is true if x_2 is true and either x_1 is true or x_3 is false

The control circuit must produce an output of 1 if at least two of the conditions A, B, and C are true. Design the simplest circuit that can be used for this purpose.

Solution: Using 1 for true and 0 for false, we can express the three conditions as follows:

$$\begin{aligned} A &= x_3(x_1 + \overline{x}_2) = x_3x_1 + x_3\overline{x}_2 \\ B &= x_1(\overline{x}_2 + \overline{x}_3) = x_1\overline{x}_2 + x_1\overline{x}_3 \\ C &= x_2(x_1 + \overline{x}_3) = x_2x_1 + x_2\overline{x}_3 \end{aligned}$$

Then, the desired output of the circuit can be expressed as $f = AB + AC + BC$. These product terms can be determined as:

$$\begin{aligned} AB &= (x_3x_1 + x_3\overline{x}_2)(x_1\overline{x}_2 + x_1\overline{x}_3) \\ &= x_3x_1x_1\overline{x}_2 + x_3x_1x_1\overline{x}_3 + x_3\overline{x}_2x_1\overline{x}_2 + x_3\overline{x}_2x_1\overline{x}_3 \\ &= x_3x_1\overline{x}_2 + 0 + x_3\overline{x}_2x_1 + 0 \\ &= x_1\overline{x}_2x_3 \end{aligned}$$

$$\begin{aligned} AC &= (x_3x_1 + x_3\overline{x}_2)(x_2x_1 + x_2\overline{x}_3) \\ &= x_3x_1x_2x_1 + x_3x_1x_2\overline{x}_3 + x_3\overline{x}_2x_2x_1 + x_3\overline{x}_2x_2\overline{x}_3 \\ &= x_3x_1x_2 + 0 + 0 + 0 \\ &= x_1x_2x_3 \end{aligned}$$

$$\begin{aligned} BC &= (x_1\overline{x}_2 + x_1\overline{x}_3)(x_2x_1 + x_2\overline{x}_3) \\ &= x_1\overline{x}_2x_2x_1 + x_1\overline{x}_2x_2\overline{x}_3 + x_1\overline{x}_3x_2x_1 + x_1\overline{x}_3x_2\overline{x}_3 \\ &= 0 + 0 + x_1\overline{x}_3x_2 + x_1\overline{x}_3x_2 \\ &= x_1x_2\overline{x}_3 \end{aligned}$$

Therefore, f can be written as

$$\begin{aligned} f &= x_1\overline{x}_2x_3 + x_1x_2x_3 + x_1x_2\overline{x}_3 \\ &= x_1(\overline{x}_2 + x_2)x_3 + x_1x_2(x_3 + \overline{x}_3) \\ &= x_1x_3 + x_1x_2 \\ &= x_1(x_3 + x_2) \end{aligned}$$

Example 2.4

Problem: Solve the problem in Example 2.3 by using Venn diagrams.

Solution: The Venn diagrams for functions A, B, and C in Example 2.3 are shown in parts a to c of Figure 2.1. Since the function f has to be true when two or more of A, B, and C are true, then the Venn diagram for f is formed by identifying the common shaded areas in the Venn diagrams for A, B, and C. Any area that is shaded in two or more of these diagrams is also shaded in f, as shown in Figure 2.1d. This diagram corresponds to the function

$$f = x_1x_2 + x_1x_3 = x_1(x_2 + x_3)$$

Example 2.5

Problem: Derive the simplest sum-of-products expression for the function

$$f = x_2\overline{x}_3x_4 + x_1x_3x_4 + x_1\overline{x}_2x_4$$

Solution: Applying the consensus property 17a to the first two terms yields

$$\begin{aligned} f &= x_2\overline{x}_3x_4 + x_1x_3x_4 + x_2x_4x_1x_4 + x_1\overline{x}_2x_4 \\ &= x_2\overline{x}_3x_4 + x_1x_3x_4 + x_1x_2x_4 + x_1\overline{x}_2x_4 \end{aligned}$$

Now, using the combining property 14a for the last two terms gives

$$f = x_2\overline{x}_3x_4 + x_1x_3x_4 + x_1x_4$$

Finally, using the absorption property 13a produces

$$f = x_2\overline{x}_3x_4 + x_1x_4$$

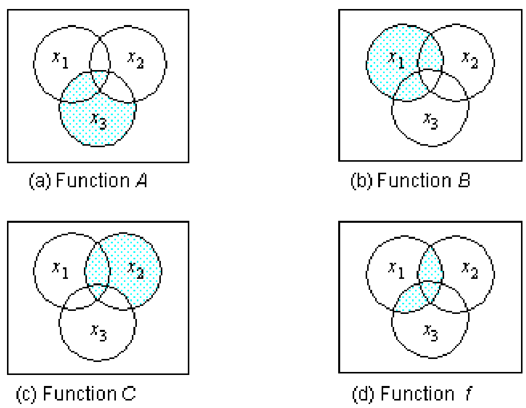

Figure 2.1. The Venn Diagrams for Example 2.4.

Example 2.6

Problem: Derive the simplest product-of-sums expression for the function

$$f = (\overline{x}_1 + x_2 + x_3)(\overline{x}_1 + \overline{x}_2 + \overline{x}_4)(\overline{x}_1 + x_3 + x_4)$$

Solution: Applying the consensus property 17b to the first two terms yields

$$\begin{aligned} f &= (\overline{x}_1 + x_2 + x_3)(\overline{x}_1 + \overline{x}_2 + \overline{x}_4)(\overline{x}_1 + x_3 + \overline{x}_1 + \overline{x}_4)(\overline{x}_1 + x_3 + x_4) \\ &= (\overline{x}_1 + x_2 + x_3)(\overline{x}_1 + \overline{x}_2 + \overline{x}_4)(\overline{x}_1 + x_3 + \overline{x}_4)(\overline{x}_1 + x_3 + x_4) \end{aligned}$$

Now, using the combining property 14b for the last two terms gives

$$f = (\overline{x}_1 + x_2 + x_3)(\overline{x}_1 + \overline{x}_2 + \overline{x}_4)(\overline{x}_1 + x_3)$$

Finally, using the absorption property 13b on the first and last terms produces

$$f = (\overline{x}_1 + \overline{x}_2 + \overline{x}_4)(\overline{x}_1 + x_3)$$