

# Generating Circuit Current Constraints to Guarantee Power Grid Safety

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**Abstract**—Efficient and early verification of the chip power distribution network is a critical step in modern IC design. Vectorless verification, developed over the last decade as an alternative to simulation based methods, requires user-specified current constraints and checks if the corresponding worst-case voltage drops at all grid nodes are below user-specified thresholds. However, obtaining/specifying the current constraints remains a burdensome task for design teams. In this paper, we define and address the *inverse* problem: for a given grid, we will generate circuit current constraints which, if adhered to by the underlying logic, would guarantee grid safety. There are many potential applications for this approach, including various grid quality metrics, as well as power grid-aware placement and floorplanning. We give a rigorous problem definition and develop some key theoretical results related to maximality of the current space defined by the constraints. Based on this, we then develop two algorithms for constraints generation that target key quality metrics like the peak total power allowed by the grid and the uniformity of the temperature distribution.

## I. INTRODUCTION

A well-designed chip power/ground network should deliver well-regulated voltages at all grid nodes in order to guarantee correct logic functionality at the intended design speed. However, with the continued scaling of semiconductor technology, there's been a corresponding increase in chip operating frequency and power dissipation. As a result, modern high-performance integrated circuits often feature large switching currents that flow in the power and ground networks, causing excessive supply voltage variations that put both circuit performance and reliability at risk. Therefore, efficient verification of power grids is a necessity in modern chip design. We will use the term "power grid" to refer to either the power or ground distribution networks. In this paper, we focus on *RC* power grids, but we are working to extend this to the *RLC* case.

Typically, power grid verification is done by simulation. The grid is simulated to determine the voltage drop at every node, assuming detailed information on the current sources tied to the grid, which represent currents drawn by the underlying circuitry. However, this method has the drawback that it requires the simulation of an exhaustive set of current waveforms in order to cover all possible scenarios and guarantee power integrity. A number of non-exhaustive methods have been proposed to implement some type of search in current space. For example, there are search techniques that find vectors to maximize the current drawn from the power network [1], as well as methods that use voltage drop analysis based on current statistics [2]. Another notable limitation of simulation based methods is that they cannot be applied at an early stage of the design flow, when detailed information on the circuit currents is not yet available.

To overcome these issues, a power grid verification scheme was

proposed in [3] that does not require full knowledge of the circuit currents. Contrary to simulation-based approaches, this method relies on information that may be available at an early stage of the design, in the form of *current constraints*. This type of approach will hence be referred to as a *vectorless* approach. Essentially, vectorless verification consists of finding the worst-case voltage fluctuations achievable at all nodes of the grid under all possible transient current waveforms that satisfy user-specified current constraints. The grid is said to be *safe* if these fluctuations are below user-specified thresholds at all grid nodes. These methods are often formulated as linear programs (LPs) to find the worst-case voltage drop under current constraints.

Several improvements have been made to mitigate the runtime complexity of vectorless verification. Because this method requires as many LPs as the number of nodes in a grid, [4] proposed a sparse approximate inverse technique to reduce the size of the LPs. Due to topological properties of the grid, voltages of grid nodes are not completely independent, which was exploited by *dominance* relations among voltage drops in [5]. Recently, in [6], a restriction of the problem to a hierarchically structured set of power constraints was considered to achieve a significant runtime improvement. Another interesting technique was proposed in [7] that verifies each subgrid of the original network independently, imposing boundary conditions on the neighboring nodes of that subgrid, i.e. without explicitly involving nodes that are not directly connected to the subgrid.

These methods have been fully developed over the last decade [8], but a key question remains: how would one obtain/specify the current constraints? In our work with colleagues in the industry, this point always comes up and remains a hurdle to adoption of these methods! The constraints are meant to capture the peak power dissipation of circuit blocks. It's easy to see how to get the constraints for a logic block that is available (down to the cell level) and small enough to exhaustively simulate, by using an "offline" characterization process. Otherwise, if the block is unavailable or too large to simulate, one must rely on engineering judgment, and/or expertise from previous design activities, which places an undue burden on users. Our work is aimed at addressing this problem.

Instead of the traditional approach of expecting users to provide current constraints that would be used to check if the grid is safe (what one might call the *forward* problem), we propose to solve the *inverse* problem: given a grid and the allowed voltage drop thresholds at all grid nodes, we will generate circuit current constraints which, if satisfied by the underlying circuitry, would guarantee grid safety. This is a significant departure from previous work and represents the first time, in our experience, that this problem has been addressed.

We believe that these current constraints would encapsulate much useful information about the grid. First of all, the availability of current constraints at an early stage of the design process provides a way to drive the rest of the design process, because these are essentially power budgets for the logic blocks under the grid. If all design teams respect these budgets throughout the design flow, then the grid is *safe by construction* at the end of the design. If the

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constraints impose too severe a budget on a certain block in some corner of the die, for example, then one can address the problem early on by modifying the grid, while it is still easy to do so, and generating a fresh set of constraints.

Alternatively, if the budgets are too severe for a candidate layout location of a high level block, then perhaps the floorplan needs to be reconsidered. Indeed, the constraints can be used to drive automated floorplanning as well as placement, so that grid-aware placement may become feasible, something that has never been done before.

Furthermore, as we will see below, our work provides a high-level and early way to *qualify* the candidate grid and assess how good it is relative to various *quality metrics*. For example, using our approach, one can check what maximum level of peak power dissipation for the whole die (or for a major part of the die) can be safely supported by the candidate power grid. If the design has a peak power budget of 100 Watts, for example, then the grid must be able to support the corresponding level of peak current in the underlying circuit, and we will see that we can verify that type of objective. Alternatively, one may be interested in spreading the power dissipation across the die in some uniform fashion, in order to avoid thermal hot-spots. We can target that objective by looking for constraints that spread the circuit current budget uniformly across the die area. Modifications of the grid may be required to allow for that, and our engine can identify whether these are needed, very early in the design process.

## II. BACKGROUND

We will review properties of a special class of matrices that will be crucial to the analysis to follow, and describe the power grid model. Throughout the rest of the paper, we will use the notation  $X \geq 0$  (or  $X > 0$ ), for any matrix  $X$ , to denote that  $X_{ij} \geq 0$  (or  $X_{ij} > 0$ ),  $\forall i, j$ , respectively. This notation will be used for vectors as well. We will also use the notation  $\mathbb{R}_+$  or  $\mathbb{R}_+^n$  to denote the non-negative part of the real space, i.e.,  $\mathbb{R}_+^n = \{x \in \mathbb{R}^n : x \geq 0\}$ . Whenever the product of a number of matrices  $A_i$  by a vector  $v$  is followed by the notation  $|_i$ , as in  $A_1 A_2 \cdots A_k v|_i$ , the expression shall denote the  $i$ th entry of the vector resulting from the product  $A_1 A_2 \cdots A_k v$ .

### A. The Class of $\mathcal{M}$ -matrices

**Definition 1.** A square matrix  $G$  is called an  $\mathcal{M}$ -matrix if:

$$g_{ij} \leq 0, \forall i \neq j \quad \text{and} \quad \Re(\lambda_i) > 0, \forall i \quad (1)$$

where  $\Re(\lambda_i)$  is the real part of the eigenvalue  $\lambda_i$ .

**Lemma 1.** [9] If  $G$  is an  $\mathcal{M}$ -matrix, then  $G^{-1}$  exists and its entries are non-negative, which we denote by  $G^{-1} \geq 0$ .

An  $n \times n$  matrix  $G$  can be used to construct a graph  $\mathcal{G}$  whose vertices are  $\{1, 2, \dots, n\}$  and whose directed edges are  $(i, j)$  for every  $g_{ij} \neq 0$ . If the graph is strongly connected (i.e. if it has a directed path from every vertex to every other vertex), then the matrix is said to be *irreducible*. A matrix  $G$  is said to be *diagonally dominant* if  $|g_{ii}| \geq \sum_{j \neq i} |g_{ij}|$ ,  $\forall i$ . A square matrix  $G$  is said to be *irreducibly diagonally dominant* if it is irreducible, it is diagonally dominant, and there is an  $i \in \{1, 2, \dots, n\}$  for which  $|g_{ii}| > \sum_{j \neq i} |g_{ij}|$ , i.e., it is *strictly diagonally dominant* in at least one row.

**Lemma 2.** [10] If  $G$  is irreducibly diagonally dominant with  $g_{ii} > 0$ ,  $\forall i$ , and  $g_{ij} \leq 0, \forall i \neq j$ , then  $G$  is an  $\mathcal{M}$ -matrix and its inverse has strictly positive entries, which we denote by  $G^{-1} > 0$ .

### B. Power Grid Model

Consider an  $RC$  model of the power grid, where every grid metal branch is represented by a resistor, and where nodes are used to represent either a via or a connection of more than two branches

on the same metal layer. We assume that there exists a capacitor from every node to ground and we ignore all line-to-line coupling capacitance. In a power grid, some nodes have ideal current sources (to ground) representing the currents drawn by the logic circuits tied to the grid at these nodes, while other nodes may be connected to ideal voltage sources representing the connection to the external voltage supply  $V_{dd}$ . Excluding the ground node, let the power grid consist of  $n+p$  nodes, where nodes  $1, 2, \dots, n$  are the nodes not connected to a voltage source, and the remaining nodes  $(n+1), (n+2), \dots, (n+p)$  are the nodes where the  $p$  voltage sources are connected. Let  $i(t)$  be the non-negative vector of all the  $m$  current sources connected to the grid, whose positive (reference) direction of current is from node-to-ground. Without loss of generality, suppose that nodes attached to current sources are numbered  $1, \dots, m$ , where  $m \leq n$ . Let  $H = [I_m \ 0]^T$  be an  $n \times m$  matrix where  $I_m$  is the  $m$ -dimensional identity matrix, and let  $i_s(t) = Hi(t)$ . Modified Nodal Analysis (MNA) [11] can be used to construct the system model [8] for the power grid:

$$Gv(t) + C\dot{v}(t) = i_s(t) \quad (2)$$

where  $v(t)$  is the  $n \times 1$  vector of time-varying voltage drops (difference between  $V_{dd}$  and true node voltages);  $C$  is the  $n \times n$  diagonal non-negative capacitance matrix, which is non-singular because every node is attached to a capacitor;  $G$  is the  $n \times n$  conductance matrix, which is known to be symmetric and diagonally dominant with positive diagonal entries and non-positive off-diagonal entries. Assuming the grid is connected (so that  $G$  is irreducible) and has at least one voltage source (so  $G$  is strictly diagonally dominant in at least one row), then  $G$  is irreducibly diagonally dominant and, by Lemma 2, we have that  $G$  is an  $\mathcal{M}$ -matrix with  $G^{-1} > 0$ .

Using a finite-difference approximation for the derivative, such as a Backward Euler numerical integration scheme  $\dot{v}(t) \approx (v(t) - v(t - \Delta t)) / \Delta t$ , the grid system model (2) leads to:

$$v(t) = A^{-1}Bv(t - \Delta t) + A^{-1}i_s(t) \quad (3)$$

where  $B = C/\Delta t$  is an  $n \times n$  diagonal matrix with  $b_{ii} > 0, \forall i$ , and  $A = G + B$ . Because  $G$  satisfies the conditions of Lemma 2, then it's clear that  $A = G + B$  also satisfies the same conditions, so that  $A$  is an  $\mathcal{M}$ -matrix with  $A^{-1} > 0$ . Let  $M = A^{-1} > 0$  and define the  $n \times m$  matrix  $M' = MH > 0$ . Furthermore, because  $B = A - G$ , then  $I_n + G^{-1}B = I_n + G^{-1}(A - G) = G^{-1}A$  where  $I_n$  is the  $n \times n$  identity matrix. But  $I_n \geq 0, G^{-1} > 0$ , and  $B \geq 0$  with  $b_{ii} > 0, \forall i$ , so that:

$$G^{-1}A > 0 \quad (4)$$

## III. PROBLEM DEFINITION

We will introduce the notion of a *container* for a vector of current waveforms, which will help us express constraints that guarantee grid safety.

**Definition 2.** (Container) Let  $t \in \mathbb{R}$ , let  $i(t) \in \mathbb{R}^m$  be a bounded function of time, and let  $\mathcal{F} \subset \mathbb{R}^m$  be a closed subset of  $\mathbb{R}^m$ . If  $i(t) \in \mathcal{F}, \forall t \in \mathbb{R}$ , then we say that  $\mathcal{F}$  contains  $i(t)$ , represented by the shorthand  $i(t) \in \mathcal{F}$ , and we refer to  $\mathcal{F}$  as a container of  $i(t)$ .

**Definition 3.** (Safe Grid) Let  $V_{th} \in \mathbb{R}^n, V_{th} \geq 0$ , be a given vector of the voltage drop thresholds at all grid nodes. We say that the grid is safe for a given  $i(t), \forall t \in \mathbb{R}$ , if the corresponding  $v(t) \leq V_{th}, \forall t \in \mathbb{R}$ .

To check if a power grid is safe, one would typically be interested in the worst-case voltage drop at some grid node  $k$ , at some time point  $\tau \in \mathbb{R}$ , over a wide range of possible current waveforms. Using the above notation, and given a container  $\mathcal{F}$  that contains a wide range of current waveforms that are of interest, we can express this

as  $\max_{i(t) \subset \mathcal{F}} (v_k(\tau))$ . Clearly, because  $\mathcal{F}$  is the same irrespective of time, and applies at all time points  $t \in \mathbb{R}$ , then this worst-case voltage drop must be time-invariant, independent of the chosen time point  $\tau$ . We now introduce the  $\text{emax}(\cdot)$  notation, which is used to capture in a single vector all the separate worst-case voltage drop maximizations, as follows.

**Definition 4.** (*emax*) For a given container  $\mathcal{F}$ , and for an arbitrary  $\tau \in \mathbb{R}$ , define:

$$v^*(\mathcal{F}) \triangleq \text{emax}_{i(t) \subset \mathcal{F}} [v(\tau)] \triangleq \begin{bmatrix} \max_{i(t) \subset \mathcal{F}} (v_1(\tau)) \\ \max_{i(t) \subset \mathcal{F}} (v_2(\tau)) \\ \vdots \\ \max_{i(t) \subset \mathcal{F}} (v_n(\tau)) \end{bmatrix} \quad (5)$$

to be the  $n \times 1$  vector whose every entry is the worst-case voltage drop at the corresponding node under all possible current waveforms  $i(t)$  contained in  $\mathcal{F}$ , with the convention that, if  $\mathcal{F} = \phi$ , then  $\text{emax}_{i(t) \subset \mathcal{F}} [v(\tau)] = 0$ .

The exact expression for  $v^*(\mathcal{F})$  was derived in [12] to be:

$$v^*(\mathcal{F}) = \sum_{q=0}^{\infty} \text{emax}_{I \in \mathcal{F}} [(A^{-1}B)^q A^{-1}I] \quad (6)$$

where  $I \in \mathbb{R}^m$  is a vector of artificial variables, with units of current, that is used to carry out the  $\text{emax}(\cdot)$  operation. One way to check node safety is by computing (6), which would then be compared against the threshold voltages. However, computing  $v^*(\mathcal{F})$  using (6) is prohibitively expensive. Instead, we will use an upper-bound on  $v^*(\mathcal{F})$  based on the following.

**Definition 5.** For any  $\mathcal{F} \subset \mathbb{R}^m$ , define:

$$\bar{v}(\mathcal{F}) \triangleq G^{-1}A \text{emax}_{I \in \mathcal{F}} (M'I) \quad (7)$$

where  $I \in \mathbb{R}^m$  is a vector of artificial variables, with units of current, that is used to carry out the  $\text{emax}(\cdot)$  operation, with the convention that  $\text{emax}_{I \in \mathcal{F}} (M'I) = 0$ , if  $\mathcal{F} = \phi$ .

In [12] and [8], the authors have derived the following upper-bound<sup>1</sup> on  $v^*(\mathcal{F})$ :

$$v^*(\mathcal{F}) \leq \bar{v}(\mathcal{F}) \quad (8)$$

**Definition 6.** (Safe Container) For a given container  $\mathcal{F}$ , we say that  $\mathcal{F}$  is safe if  $\bar{v}(\mathcal{F}) \leq V_{th}$ .

Thus, we are interested to discover a container  $\mathcal{F}$  for which  $\bar{v}(\mathcal{F}) \leq V_{th}$ , so that  $v^*(\mathcal{F}) \leq V_{th}$  and the grid is safe. We will see below that a safe container  $\mathcal{F}$  can be expressed as a set of constraints on the circuit currents that load the grid, thereby providing a set of linear constraints that are sufficient to guarantee grid safety. We will find, however, that the choice of  $\mathcal{F}$  is not unique. Indeed, there is an infinity of possible safe containers. In the following sections, we will first characterize the most desirable safe containers, and then develop algorithms to generate specific types of containers for specific objectives. In the next section, this upper bound (7) will be used to characterize the class of safe containers, which will lead us to investigate the most desirable containers, which we will call *maximal*.

<sup>1</sup>In [12], the upper bound on the worst case voltage drop is in terms of  $i(t)$  and  $M'$ , as in (7). In [8], the proof is presented in terms of  $i_s(t) = Hi(t)$  and  $M$ , but it can be easily shown that the upper bound in [8] also implies (7).

## IV. MAXIMAL CONTAINERS

This section contains the bulk of the theoretical contribution of this paper, culminating in the necessary and sufficient conditions given in Theorem 1. It lays the foundation for subsequent sections.

Let  $u \in \mathbb{R}^n$  and define the sets  $\mathcal{U}$  and  $\mathcal{F}(u)$  as follows:

$$\mathcal{U} \triangleq \{u \in \mathbb{R}^n : 0 \leq u \leq V_{th}\} \quad (9)$$

$$\mathcal{F}(u) \triangleq \{I \in \mathbb{R}^m : I \geq 0, M'I \leq MG u\} \quad (10)$$

and notice that:

$$MG u \leq MG u' \implies \mathcal{F}(u) \subseteq \mathcal{F}(u'), \quad \forall u, u' \in \mathbb{R}^n \quad (11)$$

We will see that it is enough to consider (as we will, in this paper) only containers of the form (10). The restriction to  $I \geq 0$  is obvious because  $i(t) \geq 0$  is already assumed in our grid model, above, but the rest of (10) is motivated by the following.

**Lemma 3.** For any  $u \in \mathbb{R}_+^n$ , we have  $0 \leq \bar{v}(\mathcal{F}(u)) \leq u$ .

*Proof:* For any  $u \in \mathbb{R}_+^n$ , if  $\mathcal{F}(u) = \phi$  then, from Definition 5,  $0 = \bar{v}(\mathcal{F}(u)) \leq u$ . Otherwise, if  $\mathcal{F}(u) \neq \phi$ , then Definition 5 provides that  $\bar{v}(\mathcal{F}(u)) \geq 0$ , due to (4) and the fact that  $I \geq 0$ , for all  $I \in \mathcal{F}(u)$ , by definition. Furthermore, from (10), we have  $M'I \leq MG u$ ,  $\forall I \in \mathcal{F}(u)$ , so that:

$$\text{emax}_{I \in \mathcal{F}(u)} (M'I) \leq MG u \quad (12)$$

Multiplying both sides of (12) with  $G^{-1}A \geq 0$ , due to (4), we get  $\bar{v}(\mathcal{F}(u)) \leq u$ , which completes the proof. ■

Based on this, the rest of (10) is due to the following *necessary and sufficient condition*.

**Lemma 4.** For any  $\mathcal{J} \subset \mathbb{R}_+^m$ ,  $\bar{v}(\mathcal{J}) \leq V_{th}$  if and only if  $\exists u \in \mathcal{U}$  such that  $\mathcal{J} \subseteq \mathcal{F}(u)$ .

*Proof:* The proof is in two parts.

*Proof of the “if direction:”* Let  $\mathcal{J} \subseteq \mathcal{F}(u)$  for some  $u \in \mathcal{U}$ , it follows that  $\text{emax}_{I \in \mathcal{J}} (M'I) \leq \text{emax}_{I \in \mathcal{F}(u)} (M'I)$ , from which  $\bar{v}(\mathcal{J}) \leq \bar{v}(\mathcal{F}(u))$ , due to (4). Using Lemma 3, we get  $\bar{v}(\mathcal{J}) \leq u$  which, due to  $u \in \mathcal{U}$ , gives  $\bar{v}(\mathcal{J}) \leq V_{th}$ .

*Proof of the “only if direction:”* Let  $\mathcal{J} \subset \mathbb{R}_+^m$  with  $\bar{v}(\mathcal{J}) \leq V_{th}$ , and let  $u = \bar{v}(\mathcal{J}) \leq V_{th}$ , from which:

$$G^{-1}A \text{emax}_{I \in \mathcal{J}} (M'I) = u \quad (13)$$

Because  $I \geq 0$  for any  $I \in \mathcal{J}$ , and due to (4), we have  $u \geq 0$ , so that  $u \in \mathcal{U}$ . Multiplying (13) with  $MG$ , we get:

$$\text{emax}_{I \in \mathcal{J}} (M'I) = MG u \quad (14)$$

so that,  $\forall I \in \mathcal{J}$ , we have  $M'I \leq MG u$  which, coupled with  $I \geq 0$  gives  $\mathcal{J} \subseteq \mathcal{F}(u)$ . ■

Therefore, 1)  $\mathcal{F}(u)$  is safe for any  $u \in \mathcal{U}$  and 2) all possible safe containers  $\mathcal{J}$  may be found as either specific  $\mathcal{F}(u)$  for some  $u \in \mathcal{U}$ , or as subsets of such  $\mathcal{F}(u)$ . Note that, if  $\mathcal{J} \subseteq \mathcal{F}(u)$ , for some  $u \in \mathcal{U}$ , with  $\mathcal{J} \neq \mathcal{F}(u)$ , then clearly  $\mathcal{F}(u)$  is a better choice than  $\mathcal{J}$ . Choosing  $\mathcal{J}$  would be unnecessarily limiting, while  $\mathcal{F}(u)$  would allow more flexibility in the circuit loading currents. Therefore, it is enough to consider *only* containers of the form  $\mathcal{F}(u)$  with  $u \in \mathcal{U}$ .

**Definition 7.** Define the set of safe containers:

$$\mathcal{S} \triangleq \{\mathcal{F}(u) : u \in \mathcal{U}\} \quad (15)$$

It should be clear from the above that all containers of interest are members of  $\mathcal{S}$ . Going further, if  $\mathcal{F}(u_1) \subseteq \mathcal{F}(u_2)$  with  $\mathcal{F}(u_1) \neq$

$\mathcal{F}(u_2)$ , then clearly  $\mathcal{F}(u_2)$  is a better choice than  $\mathcal{F}(u_1)$ . Thus, in a sense, the “larger” the container, the better. We capture this with the notion of *maximality*, defined as follows.

**Definition 8.** Let  $\mathcal{E}$  be a collection of subsets of  $\mathbb{R}^m$  and let  $\mathcal{X} \in \mathcal{E}$ . We say that  $\mathcal{X}$  is *maximal* in  $\mathcal{E}$  if there does not exist another  $\mathcal{Y} \in \mathcal{E}$ ,  $\mathcal{Y} \neq \mathcal{X}$ , such that  $\mathcal{X} \subseteq \mathcal{Y}$ .

Maximality is a highly desirable property and so the purpose of the rest of this section is to give necessary and sufficient conditions for a container to be maximal in  $\mathcal{S}$ . We will see that the maximality of  $\mathcal{F}(u)$  depends on crucial properties of  $u$ , as will be shown below. Note that  $0 \in \mathcal{U}$  for any  $V_{th} \geq 0$ , and  $\mathcal{F}(0) = \{0\}$ , due to  $M' > 0$  combined with  $I \geq 0$ ,  $\forall I \in \mathcal{F}(0)$ . It follows that  $\mathcal{S}$  always contains a non-empty set as a member, so that  $\mathcal{F}(u) = \phi$  is never maximal in  $\mathcal{S}$  - this will be useful below.

### A. Feasible

**Definition 9.** For any  $u \in \mathbb{R}^n$ ,  $u$  is said to be *feasible* if  $\mathcal{F}(u)$  is not empty, otherwise it is *infeasible*.

Because  $\mathcal{F}(0) = \{0\}$ , then  $u = 0$  is always feasible. In general, we have the following lemma.

**Lemma 5.** For any  $u \in \mathbb{R}^n$ ,  $u$  is feasible if and only if  $MGu \geq 0$ .

*Proof:* To prove the “if direction,” let  $u \in \mathbb{R}^n$  with  $MGu \geq 0$ , in which case clearly  $0 \in \mathcal{F}(u)$ , so that  $\mathcal{F}(u)$  is not empty and  $u$  is feasible. To prove the “only if direction,” let  $u \in \mathbb{R}^n$  be feasible so that  $\mathcal{F}(u)$  is not empty, and there exists an  $I \in \mathbb{R}^m$  such that  $I \geq 0$  and  $M'I \leq MGu$ . Due to  $M' \geq 0$  combined with  $I \geq 0$ , we have  $0 \leq M'I \leq MGu$ , so that  $MGu \geq 0$ . ■

Notice that if  $u \in \mathbb{R}^n$  is feasible then multiplying both sides of  $MGu \geq 0$  by  $G^{-1}A \geq 0$  gives  $u \geq 0$ , so that,  $u \in \mathbb{R}_+^n$ . Notice also that, if  $V_{th,k} = 0$ , then for every  $u \in \mathcal{U}$  we have  $u_k = 0$ . In this case, the only feasible  $u \in \mathcal{U}$  is  $u = 0$ , because otherwise  $MG = M(A - B) = I_n - MB$ , which leads to  $MGu|_k = u_k - MBu|_k = -MBu|_k < 0$ .

### B. Extremal

**Definition 10.** For any  $u \in \mathcal{U}$ , we say that  $u$  is *extremal* in  $\mathcal{U}$  if  $\exists k \in \{1, \dots, n\}$  such that  $u_k = V_{th,k}$ .

Denote by  $m'_{ij}$  the  $(i, j)$ th element of  $M'$  and by  $c'_j$  its  $j$ th column.

**Lemma 6.** If  $\mathcal{F}(u)$  is maximal in  $\mathcal{S}$  then  $u$  is feasible and extremal in  $\mathcal{U}$ .

*Proof:* We will prove the contrapositive. Let  $u \in \mathcal{U}$  be either infeasible or not extremal in  $\mathcal{U}$ ; we will prove that  $\mathcal{F}(u)$  is not maximal in  $\mathcal{S}$ . If  $u$  is infeasible then  $\mathcal{F}(u) = \phi$ , which we already know is not maximal in  $\mathcal{S}$ . Now consider the case when  $u$  is feasible but not extremal in  $\mathcal{U}$ . In other words, we have  $MGu \geq 0$  and  $0 \leq u < V_{th}$ , so that  $\epsilon \triangleq \min_{\forall i} (V_{th,i} - u_i) > 0$ . Let  $\mathbf{1}$  be the  $n \times 1$  vector whose every entry is 1 and let  $u' = u + \epsilon \mathbf{1}$ . Because  $G$  is irreducibly diagonally dominant with positive diagonal and non-positive off-diagonal entries, then  $G\mathbf{1} \geq 0$ , with  $G\mathbf{1} \neq 0$ . Let  $\gamma \triangleq G(u' - u) = \epsilon G\mathbf{1}$ , so that  $\gamma \geq 0$  with  $\gamma \neq 0$ , then  $M\gamma > 0$  so that  $MGu' > MGu$ . Furthermore, considering  $u' = u + \epsilon \mathbf{1}$ , we have  $0 \leq u' \leq V_{th}$  due to the definition of  $\epsilon$ , so that  $u' \in \mathcal{U}$ . We have so far established that there exists  $u' \in \mathcal{U}$  with  $MGu < MGu'$ , so that  $\mathcal{F}(u) \subsetneq \mathcal{F}(u')$ , due to (11). It only remains to prove that  $\mathcal{F}(u) \neq \mathcal{F}(u')$ . For some  $i \in \{1, \dots, m\}$ , let  $j = \operatorname{argmin}_{\forall k} (MGu'|_k / m'_{ki})$ ,  $\delta = (MGu'|_j / m'_{ji}) \geq 0$ ,  $e_i \in \mathbb{R}^m$  be the vector whose  $i$ th entry is 1 and all other entries are 0, and  $I^{(i)} = \delta e_i \geq 0$ . Notice that, for any  $k$ , we have:

$$M'I^{(i)}|_k = \delta M'e_i|_k = \delta c'_i|_k = \delta m'_{ki} \quad (16)$$

Therefore,  $M'I^{(i)}|_j = \delta m'_{ji} = MGu'|_j > MGu|_j$ , so that  $I^{(i)} \notin \mathcal{F}(u)$ . By definition of  $\delta$ , we have  $\delta \leq (MGu'|_k / m'_{ki})$ , for every  $k$ , meaning  $\delta m'_{ki} \leq MGu'|_k$ , for every  $k$ . Using (16), we then have  $M'I^{(i)}|_k = \delta m'_{ki} \leq MGu'|_k$ , for every  $k$ , which leads to  $I^{(i)} \in \mathcal{F}(u')$ , and the proof is complete. ■

### C. Irreducible

**Definition 11.** We say that  $u \in \mathbb{R}^n$  is *reducible* if there exists  $u' \leq u$ ,  $u' \neq u$ , with  $\mathcal{F}(u') = \mathcal{F}(u)$ ; otherwise,  $u$  is said to be *irreducible*.

We will see that irreducibility of  $u$  is a crucial property that is required for maximality of  $\mathcal{F}(u)$ .

**Lemma 7.** For any feasible  $u \in \mathbb{R}^n$  and any  $z \in \mathbb{R}^n$  such that  $0 \leq MGz \leq MG(u - \bar{v}(\mathcal{F}(u)))$ , let  $u' = u - z$ , it follows that  $\mathcal{F}(u') = \mathcal{F}(u)$ .

*Proof:* For any  $I \in \mathcal{F}(u')$ , we have  $I \geq 0$  and  $M'I \leq MGu' = MGu - MGz \leq MGu$ , because  $MGz \geq 0$ , so that  $I \in \mathcal{F}(u)$ . It follows that  $\mathcal{F}(u') \subseteq \mathcal{F}(u)$ . In addition, for any  $I \in \mathcal{F}(u)$ , we have  $I \geq 0$  and:

$$M'I \leq \operatorname{emax}_{I \in \mathcal{F}(u)} (M'I) = MG\bar{v}(\mathcal{F}(u)) \quad (17)$$

Notice that for any  $z$  with  $0 \leq MGz \leq MG(u - \bar{v}(\mathcal{F}(u)))$ , we have  $MGu' = MGu - MGz \geq MGu - MG(u - \bar{v}(\mathcal{F}(u))) = MG\bar{v}(\mathcal{F}(u))$ . Combining this with (17), we get  $M'I \leq MGu'$ , so that  $I \in \mathcal{F}(u')$ . Therefore,  $\mathcal{F}(u) \subseteq \mathcal{F}(u')$  from which  $\mathcal{F}(u') = \mathcal{F}(u)$ , and the proof is complete. ■

**Lemma 8.** For any feasible  $u \in \mathbb{R}^n$ , let  $u' = \bar{v}(\mathcal{F}(u))$ , it follows that  $\mathcal{F}(u') = \mathcal{F}(u)$ .

*Proof:* Let  $z = u - \bar{v}(\mathcal{F}(u))$ , so that  $MGz = MGu - MG\bar{v}(\mathcal{F}(u)) = MGu - \operatorname{emax}_{I \in \mathcal{F}(u)} (M'I) \geq 0$ , the last step due to the definition of  $\mathcal{F}(u)$ . As a result,  $z$  satisfies the conditions of Lemma 7. Let  $u' = u - z = \bar{v}(\mathcal{F}(u))$ . Then, by Lemma 7,  $\mathcal{F}(u') = \mathcal{F}(u)$ . ■

**Lemma 9.** For any  $u \in \mathbb{R}_+^n$ ,  $u$  is irreducible if and only if it is feasible and  $\bar{v}(\mathcal{F}(u)) = u$ .

*Proof:* The proof is in two parts.

*Proof of the “if direction:”* The proof is by contradiction. Let  $u$  be feasible with  $\bar{v}(\mathcal{F}(u)) = u$  and suppose that  $u$  is reducible so that there exists  $u' \leq u$ ,  $u' \neq u$ , with  $\mathcal{F}(u') = \mathcal{F}(u)$ . Notice that  $\mathcal{F}(u)$  is not empty, because  $u$  is feasible, so that  $\mathcal{F}(u')$  is not empty and  $u'$  is feasible. Therefore, we get:

$$u' - \bar{v}(\mathcal{F}(u')) = u' - \bar{v}(\mathcal{F}(u)) = u' - u + u - \bar{v}(\mathcal{F}(u))$$

Notice that, because  $u'$  is feasible, we have  $MGu' \geq 0$  from which  $u' \geq 0$ , due to (4). Because  $\bar{v}(\mathcal{F}(u')) \leq u'$ , due to Lemma 3, it follows that  $u' - u + u - \bar{v}(\mathcal{F}(u)) \geq 0$ , so that  $u - \bar{v}(\mathcal{F}(u)) \geq u - u' \geq 0$ . But  $u - u' \neq 0$ , so that  $\bar{v}(\mathcal{F}(u)) \neq u$  and we have a contradiction that completes the proof.

*Proof of the “only if direction:”* We will prove the contrapositive. Let  $u$  be either infeasible or  $\bar{v}(\mathcal{F}(u)) \neq u$ , and we will prove that  $u$  is reducible. If  $u$  is infeasible then  $\mathcal{F}(u) = \phi$  and  $u \neq 0$  (recall,  $u = 0$  is always feasible), and it is easy to find another infeasible  $u'$  with  $u' \leq u$  and  $u' \neq u$ , as follows. Let  $u' = \frac{1}{2}u$ , from which  $MGu' = \frac{1}{2}MGu \not\geq 0$ , because  $u$  is infeasible, so that  $u'$  is infeasible. Therefore, we have found  $u' \leq u$ ,  $u' \neq u$ , with  $\mathcal{F}(u') = \mathcal{F}(u) = \phi$  which means  $u$  is reducible. If  $u$  is feasible and  $\bar{v}(\mathcal{F}(u)) \neq u$ , let  $u' = \bar{v}(\mathcal{F}(u)) \in \mathbb{R}_+^n$ , due to Lemma 3, which

also provides that  $\bar{v}(\mathcal{F}(u)) \leq u$ , so that  $u' \leq u$ ,  $u' \neq u$ , with  $\mathcal{F}(u') = \mathcal{F}(u)$  due to Lemma 8, and  $u$  is reducible. ■

Note, if  $u$  is irreducible and extremal in  $\mathcal{U}$ , then  $u_k = V_{th,k}$  for some  $k$ , and  $\bar{v}(\mathcal{F}(u))|_k = V_{th,k}$ .

**Lemma 10.** For any  $u \in \mathbb{R}_+^n$ ,  $u$  is irreducible if and only if:

$$MGu \leq MGu' \iff \mathcal{F}(u) \subseteq \mathcal{F}(u'), \quad \forall u' \in \mathbb{R}^n \quad (18)$$

*Proof:* The proof is in two parts.

*Proof of the “if direction:”* We give a proof by contradiction. Given (18) and suppose  $u$  is reducible, so that it is either infeasible or  $\bar{v}(\mathcal{F}(u)) \neq u$ . If  $u$  is infeasible, then  $\mathcal{F}(u) = \phi \subseteq \mathcal{F}(u')$ , for any  $u' \in \mathbb{R}^n$ , so that  $MGu \leq MGu'$ , for any  $u' \in \mathbb{R}^n$ , due to (18). But this is impossible, because we can always find a  $u' \in \mathbb{R}^n$  that violates  $MGu \leq MGu'$ , as follows. Let  $\mathbf{1}$  be the  $n \times 1$  vector whose every entry is 1 and let  $w = -G^{-1}A\mathbf{1}$  so that  $MGw = -\mathbf{1} < 0$ , and let  $u' = u + w$  so that  $MGu' - MG u = MGw < 0$ . Therefore, it must be that  $u$  is feasible and  $\bar{v}(\mathcal{F}(u)) \neq u$ . Let  $u' = \bar{v}(\mathcal{F}(u))$ , so that  $\mathcal{F}(u') = \mathcal{F}(u)$  due to Lemma 8, with  $MGu' = MG\bar{v}(\mathcal{F}(u))$ . Recall that  $MG\bar{v}(\mathcal{F}(u)) = \text{emax}_{I \in \mathcal{F}(u)}(M'I) \leq MG u$ , and  $MG\bar{v}(\mathcal{F}(u)) \neq MG u$  due to  $\bar{v}(\mathcal{F}(u)) \neq u$ , so that  $MGu' \leq MG u$ ,  $MGu' \neq MG u$ . This means that we have  $\mathcal{F}(u) \subseteq \mathcal{F}(u')$  while  $MGu \not\leq MG u'$ , which contradicts (18), and the proof is complete.

*Proof of the “only if direction:”* Let  $u$  be irreducible, so that  $u$  is feasible with  $\bar{v}(\mathcal{F}(u)) = u$ . Due to (11), it only remains to prove that  $\forall u' \in \mathbb{R}^n, \mathcal{F}(u) \subseteq \mathcal{F}(u') \implies MG u \leq MG u'$ . Notice that  $\mathcal{F}(u')$  is non-empty, because  $\mathcal{F}(u) \neq \phi$  and  $\mathcal{F}(u) \subseteq \mathcal{F}(u')$ , from which  $u'$  is feasible. Because  $u$  and  $u'$  are feasible, and using  $u = \bar{v}(\mathcal{F}(u))$ , notice that:

$$\begin{aligned} MG u' - MG u &= MG u' - MG \bar{v}(\mathcal{F}(u)) \\ &= MG u' - \text{emax}_{I \in \mathcal{F}(u)}(M'I) \\ &\geq MG u' - \text{emax}_{I \in \mathcal{F}(u')} (M'I) \geq 0 \end{aligned}$$

where we used  $\text{emax}_{I \in \mathcal{F}(u')} (M'I) \geq \text{emax}_{I \in \mathcal{F}(u)} (M'I)$ , because  $\mathcal{F}(u) \subseteq \mathcal{F}(u')$ . Therefore,  $MG u' - MG u \geq 0$ , so  $MG u \leq MG u'$  and the proof is complete. ■

#### D. Maximality

As pointed out earlier, we are interested in safe containers that are maximal in  $\mathcal{S}$ . We now present our main result that gives the necessary and sufficient conditions for maximality.

**Theorem 1.**  $\mathcal{F}(u)$  is maximal in  $\mathcal{S}$  if and only if  $u$  is irreducible and extremal in  $\mathcal{U}$ .

*Proof:* The proof is in two parts.

*Proof of the “if direction:”* We give a proof by contradiction. Let  $u \in \mathcal{U}$  be irreducible and extremal in  $\mathcal{U}$ , but suppose that  $\mathcal{F}(u)$  is not maximal in  $\mathcal{S}$ , so that  $\exists u' \in \mathcal{U}$  such that  $\mathcal{F}(u) \subseteq \mathcal{F}(u')$ , with  $\mathcal{F}(u) \neq \mathcal{F}(u')$ . Because  $\mathcal{F}(u) \neq \mathcal{F}(u')$ , then clearly  $MG u \neq MG u'$ , and using Lemma 10, we have  $MG u \leq MG u'$ . Let  $\delta = MG u' - MG u$ , so that  $\delta \geq 0$  and  $\delta \neq 0$ . Because  $G^{-1}A > 0$  from (4), then  $G^{-1}A\delta = u' - u > 0$ . Then  $u < u' \leq V_{th}$ , due to  $u' \in \mathcal{U}$ , so that  $u$  is not extremal in  $\mathcal{U}$ , and we have a contradiction that completes the proof.

*Proof of the “only if direction:”* We give a proof by contradiction. Given that  $\mathcal{F}(u)$  is maximal in  $\mathcal{S}$ , we know from Lemma 6 that  $u$  is feasible and extremal in  $\mathcal{U}$ . Suppose  $u$  is reducible, so that  $\bar{v}(\mathcal{F}(u)) \neq u$ , because we already have that  $u$  is feasible. Recall that  $0 \leq \bar{v}(\mathcal{F}(u)) \leq u$ . Let  $u' \triangleq \bar{v}(\mathcal{F}(u)) \neq u$ , so that  $u' \in \mathcal{U}$  and  $MG u' = MG \bar{v}(\mathcal{F}(u)) = \text{emax}_{I \in \mathcal{F}(u)}(M'I)$ . Let

$\delta = MG u - MG u' = MG u - \text{emax}_{I \in \mathcal{F}(u)}(M'I)$ , then we have  $\delta \geq 0$  and  $\delta \neq 0$  (due to  $u' \neq u$ ). Because  $G^{-1}A > 0$ , then  $G^{-1}A\delta = u - u' > 0$ . Consequently, we have  $u' < u \leq V_{th}$ , due to  $u \in \mathcal{U}$ , so that  $u'$  is not extremal in  $\mathcal{U}$ . Therefore, by Lemma 6,  $\mathcal{F}(u')$  is not maximal in  $\mathcal{S}$ . However,  $\mathcal{F}(u) = \mathcal{F}(u')$ , due to Lemma 8, so that  $\mathcal{F}(u)$  is not maximal in  $\mathcal{S}$ , a contradiction that completes the proof. ■

This important theoretical result forms the basis for our choice of practical constraints generation algorithms that are guaranteed to give maximal containers, as we will see in the next section. Recall that whenever  $u$  is irreducible and extremal in  $\mathcal{U}$ , then  $\bar{v}(\mathcal{F}(u))|_k = V_{th,k}$ , for some  $k$ , so that the upper bound on the voltage drop at the  $k$ th grid node will be equal to its maximum allowable voltage drop. In other words, a maximal container always causes some node(s) on the grid to experience the maximum allowable voltage drop, at least based on the  $\bar{v}(\cdot)$  upper bound.

## V. APPLICATIONS

So far, we have shown that a container  $\mathcal{F}(u)$  is maximal in  $\mathcal{S}$  if and only if  $u$  satisfies the conditions of Theorem 1. In the following, we will discuss some design objectives that will lead us to algorithms for finding specific maximal safe containers  $\mathcal{F}(u)$ .

### A. Peak Power Dissipation

An interesting quality metric for a power grid is the peak total power dissipation that it can safely support in the underlying circuit. We refer here to the instantaneous power dissipation, which is conservatively approximated by  $V_{add} \sum_{j=1}^m i_j(t)$ . Thus, we are interested in a safe container that is maximal in  $\mathcal{S}$  and that allows the highest possible  $\sum_{j=1}^m I_j$ . For any  $u \in \mathcal{U}$ , we define  $\sigma(u)$  to be the largest sum of current source values allowed under  $\mathcal{F}(u)$ :

$$\sigma(u) \triangleq \max_{I \in \mathcal{F}(u)} \left( \sum_{j=1}^m I_j \right) \quad (19)$$

and we define  $\sigma^*$  to be the largest  $\sigma(u)$  achievable over all possible  $u \in \mathcal{U}$ , i.e.:

$$\sigma^* \triangleq \max_{u \in \mathcal{U}} (\sigma(u)) \quad (20)$$

Let  $u_p \in \mathcal{U}$  be such that  $\sigma(u_p) = \sigma^*$ , and  $I^* \in \mathcal{F}(u_p)$  be such that  $\sum_{j=1}^m I_j^* = \sigma^*$ . In general,  $u_p$  and  $I^*$  may not be unique. Based on (9) and (10), we can express the combined (19) and (20) as the following linear program (LP):

$$\begin{aligned} \sigma^* &= \text{Max} && \sum_{j=1}^m I_j \\ &\text{subject to} && M'I \leq MG u \\ &&& 0 \leq u \leq V_{th}, \quad I \geq 0 \end{aligned} \quad (21)$$

Let  $\mathcal{D}$  be the feasible region of the LP (21):

$$\mathcal{D} \triangleq \{(I, u) : I \geq 0, M'I \leq MG u, 0 \leq u \leq V_{th}\} \quad (22)$$

so that, from the above, we have:

$$\sigma^* = \max_{(I, u) \in \mathcal{D}} \left( \sum_{j=1}^m I_j \right) \quad (23)$$

Notice that,  $(0, 0) \in \mathcal{D}$  so that  $\mathcal{D}$  is not empty and all of  $\sigma^*$ ,  $u_p$ , and  $I^*$  are well-defined. Also, for every  $(I, u) \in \mathcal{D}$ , we have  $M'I \leq MG u$  and  $I \geq 0$  which, because  $M' \geq 0$ , gives  $0 \leq M'I \leq MG u$  so that  $u$  is feasible, due to Lemma 5. Therefore,  $u_p$  is feasible and the container  $\mathcal{F}(u_p) = \{I \in \mathbb{R}^m : I \geq 0, M'I \leq MG u_p\} \neq \phi$  provides the desired current constraints:

$$\begin{aligned} i(t) &\geq 0, \quad \forall t \in \mathbb{R} \\ M'i(t) &\leq MG u_p, \quad \forall t \in \mathbb{R} \end{aligned}$$

The following lemma establishes the maximality of  $\mathcal{F}(u_p)$ , based on Theorem 1. Denote by  $m_{ij}$  the  $(i, j)$ th element of  $M$  and by  $c_j$  its  $j$ th column, and notice that  $c'_j = c_j$ , for every  $j \in \{1, 2, \dots, m\}$ .

**Lemma 11.**  $\mathcal{F}(u_p)$  is maximal in  $\mathcal{S}$ .

*Proof:* We give a proof by contradiction; the proof is in two parts. First, we will prove that if  $\mathcal{F}(u_p)$  is not maximal in  $\mathcal{S}$ , then there exists a  $u \in \mathcal{U}$  that is not extremal in  $\mathcal{U}$ , with  $\sigma(u) = \sigma^*$ . Second, we will prove that, for any  $u \in \mathcal{U}$  such that  $\sigma(u) = \sigma^*$ , we have that  $u$  is extremal in  $\mathcal{U}$ , which will provide a contradiction that completes the proof.

Suppose that  $\mathcal{F}(u_p)$  is not maximal in  $\mathcal{S}$ , so that either  $u_p$  is not extremal in  $\mathcal{U}$  or  $u_p$  is reducible, due to Theorem 1. If  $u_p$  is not extremal in  $\mathcal{U}$ , then we have found a  $u = u_p \in \mathcal{U}$  that is not extremal in  $\mathcal{U}$ , with  $\sigma(u) = \sigma^*$ . If  $u_p$  is reducible, then by Lemma 9 and because  $u_p$  is feasible, we must have  $\bar{v}(\mathcal{F}(u_p)) \neq u_p$ . Let  $u' = \bar{v}(\mathcal{F}(u_p))$ , so that  $\mathcal{F}(u') = \mathcal{F}(u_p)$ , due to Lemma 8, and, by (19), we have  $\sigma(u') = \sigma(u_p) = \sigma^*$ . Let  $\delta = MG u_p - MG u'$ . Note that  $MG \bar{v}(\mathcal{F}(u_p)) = \text{emax}_{I \in \mathcal{F}(u_p)}(M'I) \leq MG u_p$  and  $MG u_p \neq MG \bar{v}(\mathcal{F}(u_p))$ , due to  $\bar{v}(\mathcal{F}(u_p)) \neq u_p$ , from which  $\delta \geq 0$  and  $\delta \neq 0$ . Combining this with  $G^{-1}A > 0$ , from (4), we have  $0 < G^{-1}A\delta = u_p - u'$ . Therefore, we have  $0 \leq u' < u_p \leq V_{th}$ , the final step due to  $u_p \in \mathcal{U}$ , so that  $u' \in \mathcal{U}$ ,  $u'$  is not extremal in  $\mathcal{U}$  and  $\sigma(u') = \sigma^*$ . This completes the first part of the proof.

Next, we will prove that, for any  $u \in \mathcal{U}$  with  $\sigma(u) = \sigma^*$ , we have that  $u$  is extremal in  $\mathcal{U}$ . For any such  $u$ , there must exist a vector  $I \in \mathcal{F}(u)$  such that  $\sum_{j=1}^m I_j = \sigma^*$ . We will proceed by contradiction. Suppose that  $u$  is not extremal in  $\mathcal{U}$ , so that  $u < V_{th}$ . Let  $\epsilon \triangleq \min_{\forall k} (V_{th,k} - u_k) > 0$ , let  $\mathbf{1}$  be the  $n \times 1$  vector whose every entry is 1, and let  $0 \leq \gamma = u + \epsilon \mathbf{1} \leq V_{th}$  due to the definition of  $\epsilon$ , from which  $\gamma \in \mathcal{U}$ . Notice that:

$$MG\gamma = MG u + \epsilon MG \mathbf{1}$$

Because  $G$  is irreducibly diagonally dominant with positive diagonal and non-positive off-diagonal entries, then  $G\mathbf{1} \geq 0$ , with  $G\mathbf{1} \neq 0$ , so that  $\epsilon MG \mathbf{1} > 0$ , because  $\epsilon M > 0$ , and  $MG\gamma > MG u$ .

Now, let  $\lambda = \min_{\forall i} (MG\gamma|_i - MG u|_i) / \max_{\forall i,j} (m_{ij})$ . Because  $MG\gamma > MG u$  and  $M > 0$ , it follows that  $\lambda > 0$ . Also, let  $e_1 \in \mathbb{R}^n$  be the vector whose 1st entry is 1 and all other entries are 0 and let  $I' = I + \lambda e_1$ . Because  $\lambda > 0$ , we have  $\lambda e_1 \geq 0$ ,  $\lambda e_1 \neq 0$ ,  $I' \geq I \geq 0$ , and  $I' \neq I$ , so that  $\sum_{j=1}^m I'_j > \sum_{j=1}^m I_j = \sigma^*$ . Furthermore, we have  $I' \in \mathcal{F}(\gamma)$ , because:

$$M'I' = M'I + \lambda M'e_1 = M'I + \lambda c'_1 \quad (24)$$

$$= M'I + \frac{\min_{\forall i} (MG\gamma|_i - MG u|_i)}{\max_{\forall i,j} (m_{ij})} c_1 \quad (25)$$

$$\leq MG u + \min_{\forall i} (MG\gamma|_i - MG u|_i) \mathbf{1} \quad (26)$$

$$\leq MG\gamma \quad (27)$$

where in (26) we used  $I \in \mathcal{F}(u)$  and  $c_1 / \max_{\forall i,j} (m_{ij}) \leq \mathbf{1}$ . Therefore, we have found  $\gamma \in \mathcal{U}$  and  $I' \in \mathcal{F}(\gamma)$  with  $\sigma(\gamma) \geq \sum_{j=1}^m I'_j > \sigma^*$ , which contradicts (23). It follows that  $u$  is extremal in  $\mathcal{U}$ . ■

As an example, the LP (21) is run on the small grid in Fig. 1 and the resulting container is shown in Fig. 2 where  $u_p = [89 \ 100 \ 95 \ 98]^T$  (units of mV). Notice that this method, in order to allow the maximum peak power, may generate a container that is skewed in a way that imposes a tight constraint on current in certain locations of the die (such as at  $i_2(t)$ ) while allowing larger current in other locations (such as at  $i_1(t)$ ). Other approaches are possible to avoid this skew and even out the current budgets, as we will see next.

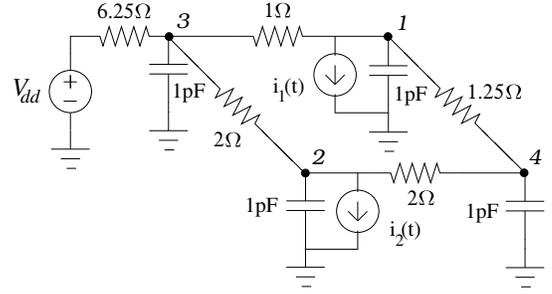


Fig. 1: An example of a power grid with 4 nodes, 2 current sources, and  $V_{th} = [110 \ 100 \ 95 \ 105]^T$  (units of mV).

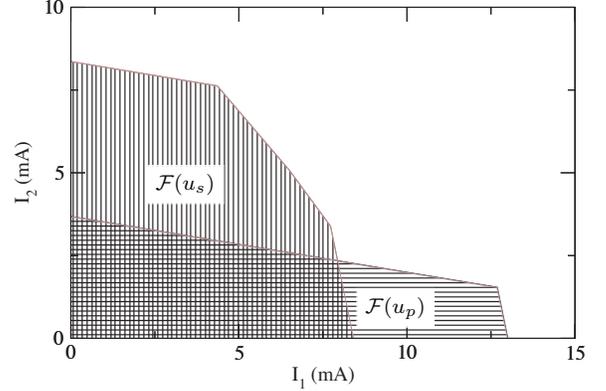


Fig. 2: An example of  $\mathcal{F}(u_p)$  and  $\mathcal{F}(u_s)$ .

## B. Uniform Current Distribution

The design team may be interested in a grid that safely supports a uniform current distribution across the die, so as to allow a placement that provides a uniform temperature distribution. We can generate constraints that allow that objective by searching for a safe maximal container  $\mathcal{F}(u)$  that contains the hypersphere in current space that has the largest volume, or the largest radius  $\theta$ . In other words, this method aims to “raise the minimum” and avoid the skew indicated above. We will develop a method (34) which, when applied to the simple grid in Fig. 1, generates the container  $\mathcal{F}(u_s)$  shown in Fig. 2, where  $u_s = [83 \ 91 \ 95 \ 92]^T$  (units of mV).

Let  $S(\theta) \subset \mathbb{R}^m$  denote the hypersphere with radius  $\theta$ , centered at the origin and let  $S^+(\theta) = S(\theta) \cap \mathbb{R}_+^m$  be the part of that hypersphere that is in the first quadrant of  $\mathbb{R}^m$ . Denote by  $r_i$  and  $r'_i$  the  $i$ th rows of  $M$  and  $M'$ , respectively. For any  $u \in \mathcal{U}$ , define  $\mathcal{H}_i = \{I \in \mathbb{R}^m : I \geq 0, r'_i I = r_i G u\}$  to be the hyperplane that constitutes the  $i$ th outer boundary of  $\mathcal{F}(u)$ . Define  $D_i$  to be the distance from the origin to  $\mathcal{H}_i$  which, according to [13], can be expressed as:

$$D_i = \frac{|r_i G u|}{d_i} \quad (28)$$

where  $d_i = \sqrt{\sum_{j=1}^m m_{ij}^2} > 0$ . As we're interested in a non-empty  $\mathcal{F}(u)$ , we will enforce that  $\theta \geq 0$  and  $u$  is feasible, i.e.,  $r_i G u \geq 0$ ,  $\forall i$ , so that:

$$D_i = \frac{r_i G u}{d_i} \quad (29)$$

In order to have  $S^+(\theta) \subseteq \mathcal{F}(u)$ , we will require that:

$$\theta \leq D_i, \quad \forall i \in \{1, \dots, n\} \quad (30)$$

TABLE I: Results of both LPs

Power Grid				Peak Power		Uniform Current Distribution		Peak Power using $u_s$	Uniform Current Dist. using $u_p$
Name	Nodes	Current Sources	SPAI Time	$P(u_p)$ in mW	CPU Time	$\rho(u_s)$ in $\mu A$	CPU Time	$P(u_s)$ in mW	$\rho(u_p)$ in $\mu A$
G1	8,413	552	1.3 min	1.73	1.15 min	1.77	2.2 min	0.6	0.84
G2	18,678	1,119	4.6 min	3.71	5.51 min	2.11	12.86 min	1.86	1.06
G3	32,554	2,070	13.38 min	6.75	23.51 min	1.47	36.7 min	3.02	0.462
G4	50,444	3,192	27.7 min	9.83	29.28 min	1.8	57.2 min	5.09	0.8
G5	113,304	7,140	2.52 h	22.06	1.92 h	1.44	3.38 h	11.07	0.45
G6	200,828	12,656	6.8 h	40.69	6.37 h	2.11	11.9 h	22.7	0.98
G7	312,232	19,460	17.73 h	59.77	11.19 h	2.03	18.68 h	35.23	0.98

which can be expressed compactly as:

$$\theta d \leq MG u \quad (31)$$

where  $d = [d_1 \ \dots \ d_n]^T$ . For any  $u \in \mathcal{U}$ , we define  $\rho(u)$  to be the largest  $\theta \geq 0$  for which (31) is satisfied, so that:

$$\rho(u) \triangleq \max_{0 \leq \theta d \leq MG u} (\theta) \quad (32)$$

and we define  $\rho^*$  to be the largest  $\rho(u)$  achievable over all possible  $u \in \mathcal{U}$ , i.e.:

$$\rho^* \triangleq \max_{u \in \mathcal{U}} (\rho(u)) \quad (33)$$

Let  $u_s \in \mathcal{U}$  be such that  $\rho(u_s) = \rho^*$ . In general,  $u_s$  may not be unique. We can express the combined (32) and (33) as the following linear program:

$$\begin{aligned} \rho^* = \quad & \text{Max} && \theta \\ \text{subject to} && \theta d \leq MG u \\ && 0 \leq u \leq V_{th}, \quad \theta \geq 0 \end{aligned} \quad (34)$$

Let  $\mathcal{R}$  be the feasible region of the LP (34):

$$\mathcal{R} \triangleq \{(\theta, u) : \theta \geq 0, \theta d \leq MG u, 0 \leq u \leq V_{th}\} \quad (35)$$

so that, from the above, we have:

$$\rho^* = \max_{(\theta, u) \in \mathcal{R}} (\theta) \quad (36)$$

Notice that,  $(0, 0) \in \mathcal{R}$  so that  $\mathcal{R}$  is not empty and  $\rho^*$  and  $u_s$  are well-defined. Also, for every  $(\theta, u) \in \mathcal{R}$ , we have  $\theta d \leq MG u$  and  $\theta \geq 0$ . Because  $d \geq 0$ , it follows that  $0 \leq \theta d \leq MG u$  so that  $u$  is feasible, due to Lemma 5. Therefore,  $u_s$  is feasible and the container  $\mathcal{F}(u_s) = \{I \in \mathbb{R}^m : I \geq 0, M'I \leq MG u_s\} \neq \emptyset$  provides the desired current constraints:

$$\begin{aligned} i(t) &\geq 0, \quad \forall t \in \mathbb{R} \\ M'i(t) &\leq MG u_s, \quad \forall t \in \mathbb{R} \end{aligned}$$

The following lemma, based on Theorem 1, establishes the maximality of  $\mathcal{F}(u_s)$ .

**Lemma 12.**  $\mathcal{F}(u_s)$  is maximal in  $\mathcal{S}$ .

*Proof:* We give a proof by contradiction; the proof is in two parts. First, we will prove that if  $\mathcal{F}(u_s)$  is not maximal in  $\mathcal{S}$ , then there exists a  $u \in \mathcal{U}$  that is not extremal in  $\mathcal{U}$ , with  $\rho(u) = \rho^*$ . Second, we will prove that, for any  $u \in \mathcal{U}$  such that  $\rho(u) = \rho^*$ , we have that  $u$  is extremal in  $\mathcal{U}$ , which will provide a contradiction that completes the proof.

Suppose that  $\mathcal{F}(u_s)$  is not maximal in  $\mathcal{S}$ , so that either  $u_s$  is not extremal in  $\mathcal{U}$  or  $u_s$  is reducible, due to Theorem 1. If  $u_s$  is not extremal in  $\mathcal{U}$ , then we have found a  $u = u_s \in \mathcal{U}$  that is not extremal in  $\mathcal{U}$ , with  $\rho(u) = \rho^*$ . If  $u_s$  is reducible, then by Lemma 9 and because  $u_s$  is feasible, we must have  $\bar{v}(\mathcal{F}(u_s)) \neq u_s$ . Let  $u' = \bar{v}(\mathcal{F}(u_s))$ , so that  $\mathcal{F}(u') = \mathcal{F}(u_s)$  due to Lemma 8. Clearly,  $S^+(\rho^*) \subseteq \mathcal{F}(u')$ , because  $S^+(\rho^*) \subseteq \mathcal{F}(u_s)$ , so that  $\rho(u') = \rho^*$ . Let  $\delta = MG u_s - MG u'$ . Note that  $MG \bar{v}(\mathcal{F}(u_s)) =$

$\text{emax}_{I \in \mathcal{F}(u_s)}(M'I) \leq MG u_s$  and  $MG u_s \neq MG \bar{v}(\mathcal{F}(u_s))$ , due to  $\bar{v}(\mathcal{F}(u_s)) \neq u_s$ , from which  $\delta \geq 0$  and  $\delta \neq 0$ . Combining this with  $G^{-1}A > 0$ , from (4), we have  $G^{-1}A\delta = u_s - u' > 0$ . Therefore, we have  $0 \leq u' < u_s \leq V_{th}$ , the final step due to  $u_s \in \mathcal{U}$ , so that  $u' \in \mathcal{U}$ ,  $u'$  is not extremal in  $\mathcal{U}$  and  $\rho(u') = \rho^*$ . This completes the first part of the proof.

Next, we will prove that, for any  $u \in \mathcal{U}$  with  $\rho(u) = \rho^*$ , we have that  $u$  is extremal in  $\mathcal{U}$ . We will proceed by contradiction. Suppose that  $u$  is not extremal in  $\mathcal{U}$ , so that  $u < V_{th}$ . Let  $\epsilon \triangleq \min(V_{th,k} - u_k) > 0$ , let  $\mathbf{1}$  be the  $n \times 1$  vector whose every entry is 1, and let  $0 \leq \gamma = u + \epsilon \mathbf{1} \leq V_{th}$ , due to the definition of  $\epsilon$ , from which  $\gamma \in \mathcal{U}$ . Notice that:

$$MG \gamma = MG u + \epsilon MG \mathbf{1} \quad (37)$$

Because  $G$  is irreducibly diagonally dominant with positive diagonal and non-positive off-diagonal entries, then  $G \mathbf{1} \geq 0$ , with  $G \mathbf{1} \neq 0$ , so that  $\epsilon MG \mathbf{1} > 0$ , because  $\epsilon M > 0$ , and  $MG \gamma > MG u$ .

Now, let  $\lambda = \min_{\forall i} (MG \gamma|_i - MG u|_i) / \max_{\forall i} (d_i)$  and let  $\theta' = \rho^* + \lambda$ . Because  $MG \gamma > MG u$  and  $d_i > 0, \forall i$ , it follows that  $\lambda > 0$  and  $\theta' > \rho^* \geq 0$ . Furthermore, we have  $(\theta', \gamma) \in \mathcal{R}$ , because:

$$\theta' d = \rho^* d + \frac{\min_{\forall i} (MG \gamma|_i - MG u|_i)}{\max_{\forall i} (d_i)} d \quad (38)$$

$$\leq MG u + \min_{\forall i} (MG \gamma|_i - MG u|_i) \mathbf{1} \quad (39)$$

$$\leq MG \gamma \quad (40)$$

where in (39) we used  $(\rho^*, u) \in \mathcal{R}$  and  $d / \max_{\forall i} (d_i) \leq \mathbf{1}$ . Therefore, we have found  $(\theta', \gamma) \in \mathcal{R}$  and  $\theta' > \rho^*$ , which contradicts (36). It follows that  $u$  is extremal in  $\mathcal{U}$ . ■

## VI. RESULTS

The above two algorithms (21) and (34) have been implemented using C++. Both problems require the computation of the inverse of  $A$ , which was generated using the sparse approximate inverse technique (SPAI), as was done in [4]. The maximizations were performed using the Mosek optimization package [14]. We conducted tests on a set of power grids with a 1.1 V supply voltage that were generated based on user-specifications, including grid dimensions, metal layers, pitch and width per layer, and C4 and current source distributions, consistent with 65nm technology. All results were obtained using a 2.6 GHz Linux machine with 24 GB of RAM.

The number of variables in (21) is  $n + m$ , and the number of variables in (34) is  $n + 1$ , where  $n$  is the total number of nodes and  $m$  is the number of current sources attached to the grid. Denote by  $P(u) \triangleq V_{dd} \times \sigma(u)$  the peak power dissipation allowed under  $\mathcal{F}(u)$ . In Table I, we present the results of both LPs in columns 5 and 7, respectively. For instance, on a 312,232 node grid, the peak power dissipation is 59.77 mW and the largest current radius for which the part of the hypersphere in the first quadrant is contained in  $\mathcal{F}(u_s)$  is 2.03  $\mu A$ . The CPU times for solving (21) and (34) are given in columns 6 and 8, respectively. Note that these CPU times do not include the time for computing the approximate inverse, which is

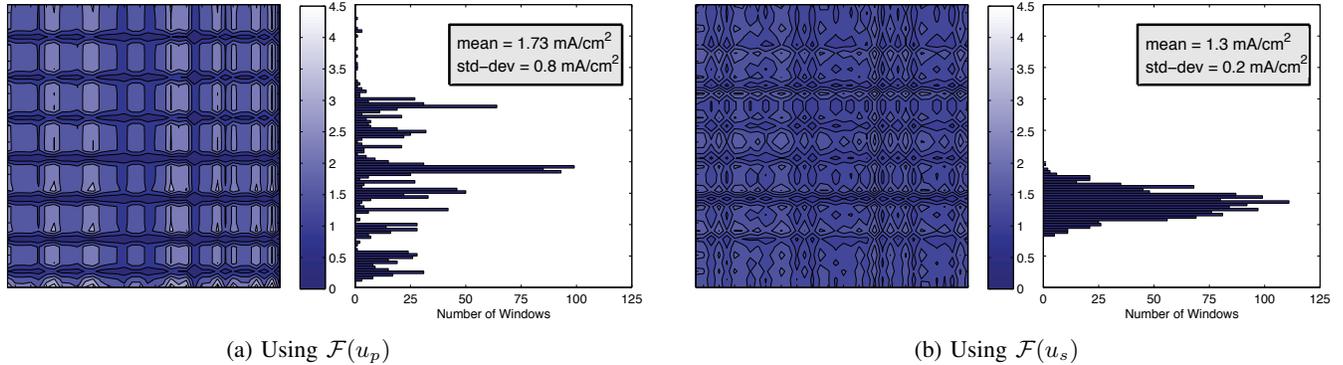


Fig. 3: Contour plots for peak power density across the layout and the corresponding histograms. The color bar units are  $\text{mA}/\text{cm}^2$ .

reported separately in column 4. For example, the total CPU time for solving (21) corresponds to the sum of the CPU times reported in columns 4 and 6. The only source of error is the sparse approximation (SPAD) of  $A^{-1}$ , which is controlled by enforcing an error tolerance of  $10^{-4}$  between every entry of the exact inverse and the corresponding entry of the approximate inverse.

To study the difference between the containers generated using (21) and (34), we used two methods. First, we computed the peak power dissipation achievable under  $\mathcal{F}(u_s)$ , which is  $P(u_s)$ , and the largest current radius for which the part of the hypersphere in the first quadrant is contained in  $\mathcal{F}(u_p)$ , which is  $\rho(u_p)$ . The results obtained for  $P(u_s)$  and  $\rho(u_p)$  are reported in columns 9 and 10 of Table I. The results show that  $P(u_s) \ll P(u_p)$  and  $\rho(u_p) \ll \rho(u_s)$  on all grids. In fact, the peak power dissipation achievable under  $\mathcal{F}(u_s)$  is at most 59% of that achievable under  $\mathcal{F}(u_p)$ . Also, the largest current radius for which the part of the hypersphere in the first quadrant is contained in  $\mathcal{F}(u_p)$  is at most 50% of that contained in  $\mathcal{F}(u_s)$ . Thus, each approach provides a distinct trade-off for the chip design team.

Another way to compare the two approaches (21) and (34), is to look at the *power density*, i.e., the power dissipation per unit area of the die, allowed by the two resulting containers. To assess this, we maximize the allowed power (current) within a small window of the die surface, and we do this for every position of that window across the die. We divide the die area into  $\kappa \times \kappa$  of these windows and compute the peak power density inside each, as allowed by  $\mathcal{F}(u_p)$  or  $\mathcal{F}(u_s)$ . In Figs. 3a and 3b, we present contour plots for  $\kappa = 35$  for the peak power densities under  $\mathcal{F}(u_p)$  and  $\mathcal{F}(u_s)$ , respectively, on a 50k node grid. Note that the current constraints based on  $\mathcal{F}(u_p)$  allow higher current densities at certain spots but also include some spots with very small and restricted current density budgets. This large spread in power densities can lead to thermal hotspots. This may be avoided by using  $\mathcal{F}(u_s)$  which, as expected and as seen in the figure, provides a uniform distribution of power densities across the die area compared to  $\mathcal{F}(u_p)$ , which is reflected in a smaller standard deviation. Of course,  $\mathcal{F}(u_p)$  supports a larger overall peak power dissipation than  $\mathcal{F}(u_s)$ , which is reflected in a larger mean. There is a clear trade-off between the two methods, making them both useful but also pointing the way to future work for managing this trade-off and investigating other possibilities.

## VII. CONCLUSION

Early power grid verification is a key step in modern chip design. Traditionally, it has been performed either by simulation or by vectorless verification, both of which have serious shortcomings. We propose a novel method to solve the *inverse* problem of vectorless verification, by generating circuit current constraints that ensure power grid safety. We develop some key theoretical results to allow the

generation of constraints that correspond to maximal current spaces. We then apply these results to provide two constraints generation algorithms that target key quality metrics of the grid: the maximum power dissipation the grid can safely support and the uniformity of the power spread across the die.

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