Generating Current Budgets to Guarantee Power Grid Safety

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***Abstract*—Efficient and early verification of the chip power distribution network is a critical step in modern chip design. Vectorless verification, developed over the last decade as an alternative to simulation-based methods, requires user-specified current constraints (budgets) and checks if the corresponding worst-case voltage drops at all grid nodes are below user-specified thresholds. However, obtaining/specifying the current constraints remains a burdensome task for users. In this paper, we define and address the inverse problem: for a given grid, we would like to generate circuit current constraints which, if adhered to by the underlying logic, would guarantee grid safety. There are many potential applications for this approach, including various grid quality metrics, as well as voltage drop aware placement and floorplanning. We give a rigorous problem definition and develop some key theoretical results related to maximality of the current space defined by the constraints. Based on this, we then develop two algorithms for constraints generation that target the peak total chip power that is allowed by the grid and the uniformity of the temperature distribution. Finally, we develop a superior algorithm which targets a combination of both quality metrics.**

***Index Terms*—Current constraints, integrated circuits, optimization, power grid, verification.**

I. Introduction

A WELL-DESIGNED power/ground network of an integrated circuit should deliver well-regulated voltages at all supply nodes in order to guarantee correct logic functionality at the intended design speed. However, with the continued scaling of semiconductor technology, there has been a corresponding increase in chip power dissipation, along with a reduction of the supply voltage. As a result, modern high-performance integrated circuits often feature large switching currents that flow in the power and ground networks, causing excessive supply voltage variations that put both circuit performance and reliability at risk. Therefore, efficient verification of power grids is a necessity in modern chip design. We will use the term power grid to refer to either the power or the ground distribution networks. In this paper, we focus on *RC* power grids, but we are working to extend this to the *RLC* case.

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Typically, power grid verification is done by simulation. The grid is simulated to determine the voltage drop at every node, given detailed information on the current sources tied to the grid, which represent currents drawn by the underlying circuitry. However, this method has the drawback that it requires the simulation of an exhaustive set of current waveforms in order to cover all possible scenarios and guarantee power integrity. A number of nonexhaustive methods have been proposed which implement some type of search in current space. For example, there are search techniques that find vectors to maximize the current drawn from the power network [1], as well as methods that use voltage drop analysis based on current statistics [2]. However, search-based methods are often not scalable to large designs, and while stochastic methods offer some confidence, they do not offer a guarantee that the design is safe from voltage-drop violations.

An alternative power grid verification scheme was proposed in [3] that is not simulation based. It does not require full knowledge of the circuit currents, but relies on information that may be available at an early stage of the design, in the form of current budgets or current constraints. This offers the added advantage that it is applicable very early in the design flow, before full details of the circuitry under the grid is known. This type of approach will hence be referred to as a vectorless approach. Essentially, vectorless verification consists of finding the worst-case voltage fluctuations achievable at all nodes of the grid under all possible transient current waveforms that satisfy user-specified current constraints. The grid is said to be safe if these fluctuations are below user-specified thresholds at all grid nodes. These methods are solved as linear programs (LPs) that can become expensive if applied to the whole grid.

Several improvements have been made to mitigate the runtime complexity of vectorless verification. For example, Ghani and Najm [4] proposed a sparse approximate inverse (SPAI) technique to reduce the size of the LPs. As well, dominance relations among voltage drops of grid nodes were exploited in [5] to reduce the computational workload. Recently, in [6], a restriction of the problem to a hierarchically structured set of power constraints was considered, leading to significant runtime improvement. Another interesting technique was proposed in [7] that verifies each subgrid of the original network independently, imposing boundary conditions on the neighboring nodes of that subgrid, i.e., without explicitly involving nodes that are not directly connected to the subgrid.

These methods have been fully developed over the last decade [8], but a key question remains: how would one obtain/specify the current constraints? In our work with industry colleagues, this point always comes up and is a hurdle to adoption of these methods! The constraints are meant to capture the peak power dissipation of circuit blocks. It is easy to see how to get the constraints for a logic block that is available (down to the cell level) and small enough to exhaustively simulate, by using an "offline" characterization process. Otherwise, if the block is unavailable or too large to simulate, one must rely on engineering judgment, and/or expertise from previous design activities, which seems to place an undue burden on users. This paper is aimed at addressing this problem.

Instead of the traditional approach of expecting users to provide current constraints that would be used to check if the grid is safe (what one might call the forward problem), we propose to solve the inverse problem: given a grid and the allowed voltage drop thresholds at all grid nodes, we would like to generate circuit current constraints which, if satisfied by the underlying circuitry, would guarantee grid safety. This is a significant departure from previous work and represents the first time that this problem has been addressed.

We believe that these current constraints would encapsulate much useful information about the grid. First of all, the availability of current constraints at an early stage of the design flow provides a way to drive the rest of the design process, because these are essentially power budgets for the logic blocks under the grid. If all design teams respect these budgets throughout the design flow, then the grid is safe by construction at the end. If the constraints impose too severe a budget on a certain block in some location on the die, then one can address the problem early on by modifying the grid while it is still easy to do so and/or by generating a fresh set of constraints. Effectively, this would give a systematic automated process for reallocating the block power budgets across the die, a rebudgeting process that is done mostly manually today. Alternatively, the floorplan may need to be reconsidered and, in fact, the constraints may be used to drive automated floorplanning or placement, so that grid-aware placement may become feasible, something that has never been done before. In addition, during the late stages of the design, as low-level blocks are being refined and optimized, if there is a need to check the safety of certain circuit switching scenarios, this may be done by simply rechecking the resulting current traces against the current constraints, without having to resort to any grid simulation.

Furthermore, as we will see below, this paper provides a high-level and early way to qualify the candidate grid and assess how good it is relative to various quality metrics. For example, using our approach, one can check what maximum level of peak power dissipation for the whole die (or for a major part of the die) can be safely supported by the candidate power grid. If the design has a peak power budget of 100 W, for example, then the grid must be able to support the corresponding level of peak current in the underlying circuit (without any voltage drop violations), and we will see that we can verify that type of objective. Alternatively, one may be interested in spreading the power dissipation across the die in some uniform fashion, in order to avoid thermal hot spots. We can target that objective by looking for constraints that spread the circuit current budgets uniformly across the die area. Modifications of the grid may be required to allow for that, and our engine can identify whether these are needed, very early in the design process.

A preliminary version of this paper has appeared in [9]. The rest of this paper is organized as follows. The next section gives a brief overview of a special class of matrices and describes the power grid model. In Section III, we give the problem definition and review some key results from previous work. We then present the highly desirable property of maximality of the current spaces, along with our main theoretical contribution, in Section IV. In Section V, we give three algorithms for generating circuit current constraints that are provably maximal. Section VI presents some test results from our implementation of these methods and describes the tradeoffs among the three algorithms. Finally, Section VII gives some concluding remarks.

II. Background

We will review a few properties of a special class of matrices that will be crucial to the analysis to follow and describe the power grid model. Throughout this paper, we will use the notation $x \leq y$ (or $x < y$), for any two vectors x and y, to denote that $x_i \leq y_i$ (or $x_i < y_i$), $\forall i$, respectively. Similarly, we will use the notation $X \geq 0$ (or $X > 0$), for any matrix X, to denote that $X_{ij} \geq 0$ (or $X_{ij} > 0$), $\forall i, j$, respectively. We will also use the notation $\mathbb{R}_+$ or $\mathbb{R}^n_+$ to denote the non-negative part of the real space, i.e., $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x \geq 0\}$. Whenever the product of a number of matrices A_i by a vector v is followed by the notation $|_i$, as in $A_1 A_2 \cdots A_k v|_i$, the expression shall denote the ith entry of the vector resulting from the product $A_1 A_2 \cdots A_k v$.

A. Class of $\mathcal{M}$-Matrices

We use standard definitions and results from [10] and [11].

Definition 1: A square matrix G is called an $\mathcal{M}$-matrix if

$$g_{ij} \leq 0, \ \forall i \neq j \qquad \text{and} \qquad \Re(\lambda_i) > 0, \ \forall i \tag{1}$$

where $\Re(\lambda_i)$ is the real part of the eigenvalue λ_i of G.

Lemma 1 [10]: If G is an $\mathcal{M}$-matrix, then G^{-1} exists and all its entries are non-negative, i.e., $G^{-1} \geq 0$.

An $n \times n$ matrix G can be used to construct a graph $\mathcal{G}$ whose vertices are $\{1, 2, \ldots, n\}$ and whose directed edges are (i, j) for every $g_{ij} \neq 0$. If the graph is strongly connected (i.e., if it has a directed path from every vertex to every other vertex), then the matrix is said to be irreducible. A matrix G is said to be diagonally dominant if $|g_{ii}| \geq \sum_{j \neq i} |g_{ij}|, \ \forall i$. A square matrix G is said to be irreducibly diagonally dominant if it is irreducible, it is diagonally dominant, and there is an $i \in \{1, 2, \ldots, n\}$ for which $|g_{ii}| > \sum_{j \neq i} |g_{ij}|$, i.e., it is strictly diagonally dominant in at least one row.

Lemma 2 [11]: If G is irreducibly diagonally dominant with $g_{ii} > 0, \forall i$, and $g_{ij} \leq 0, \forall i \neq j$, then G is an $\mathcal{M}$-matrix and its inverse has strictly positive entries, i.e., $G^{-1} > 0$.

B. Power Grid Model

Consider an *RC* model of the power grid, where every grid metal branch is represented by a resistor, and where nodes are used to represent either a via or a connection of more than two branches on the same metal layer. We assume that there exists a capacitor from every node to ground and we ignore all line-to-line coupling capacitance. At its metal-1 terminals, the grid is loaded by the circuit blocks, where nonlinearities are encountered due to the circuit MOSFETs. It is practically impossible to jointly simulate or analyze both the full nonlinear circuit and the large grid all at once, and common practice is to decouple the two. This typically means that the circuit blocks are represented by some suitable model, consisting of a current source along with some parasitic network to ground. However, for grid verification, these parasitics are often neglected because of the larger impact that uncertainty of the currents has on the grid response, and so the circuit current sources are often assumed ideal—and this is what will be assumed in this paper. However, this is not a limitation of this paper. In fact, with the addition of *RC* parasitics to ground to every current source, for example, the properties of the circuit matrices would improve and they become a little bit easier to solve, but for now we will keep current source parasitics out of the picture for clarity of the presentation.

Therefore, in the power grid model used in this paper, some nodes have ideal current sources (to ground) representing the currents drawn by the logic circuits tied to the grid at these nodes, while other nodes may be connected to ideal voltage sources representing the connection to the external voltage supply V_{dd}. Excluding the ground node, let the power grid consist of $n+p$ nodes, where nodes $1, 2, \ldots, n$ are the nodes not connected to a voltage source, and the remaining nodes $(n+1), (n+2), \ldots, (n+p)$ are the nodes where the p voltage sources are connected. Let $i(t)$ be the non-negative vector of all the m current sources connected to the grid, whose positive (reference) direction of current is from node-to-ground. Without loss of generality, suppose that nodes attached to current sources are numbered $1, \ldots, m$, where $m \leq n$. Let $H = \begin{bmatrix} I_m & 0 \end{bmatrix}^T$ be an $n \times m$ matrix where I_m is the m-dimensional identity matrix, and let $i_s(t) = Hi(t)$.

Let $\hat{v}(t)$ be the vector of node voltages, relative to ground. By superposition, $\hat{v}(t)$ may be found in three steps: 1) open-circuit all the current sources and find the response, which would obviously be $\hat{v}^{(1)}(t) = V_{\mathrm{dd}}$ in this case; 2) short-circuit all the voltage sources and find the response $\hat{v}^{(2)}(t)$, in this case $\hat{v}^{(2)}(t) \leq 0$; and 3) find $\hat{v}(t) = \hat{v}^{(1)}(t) + \hat{v}^{(2)}(t)$. To find $\hat{v}^{(2)}(t)$, KCL at every node easily provides, via nodal analysis [12], that

$$G\hat{v}^{(2)}(t) + C\hat{v}'^{(2)}(t) = -i_s(t)$$

where C is the $n \times n$ diagonal non-negative capacitance matrix, which is nonsingular because every node is attached to a capacitor; G is the $n \times n$ conductance matrix, which is known to be symmetric and diagonally dominant with positive diagonal entries and nonpositive off-diagonal entries. Assuming the graph consisting of all grid nodes $1, 2, \cdots, n$ and all grid resistances in between these nodes is a connected graph (so that G is irreducible) and has at least one voltage source (so that G is strictly diagonally dominant in at least one row), then G is irreducibly diagonally dominant and, by Lemma 2, we have that G is an $\mathcal{M}$-matrix with $G^{-1} > 0$. We are mainly interested in the voltage drop $v(t) = V_{\mathrm{dd}} - \hat{v}(t) = -\hat{v}^{(2)}(t) \geq 0$, rather than $\hat{v}(t)$, so that

$$Gv(t) + C\dot{v}(t) = i_s(t) \tag{2}$$

where $v(t)$ is the $n \times 1$ vector of time-varying voltage drops (difference between V_{dd} and true node voltages).

Using a finite-difference approximation for the derivative, such as a Backward Euler numerical integration scheme $\dot{v}(t) \approx (v(t) - v(t - \Delta t))/\Delta t$, the grid system model (2) leads to

$$v(t) = A^{-1}Bv(t - \Delta t) + A^{-1}i_s(t) \tag{3}$$

where $B = C/\Delta t$ is an $n \times n$ diagonal matrix with $b_{ii} > 0$, $\forall i$, and $A = G + B$. Because G satisfies the conditions of Lemma 2, then it is clear that $A = G + B$ also satisfies the same conditions, so that A is an $\mathcal{M}$-matrix with $A^{-1} > 0$. Let $M = A^{-1} > 0$ and define the $n \times m$ matrix $M' = MH > 0$. Furthermore, because $B = A - G$, then $I_n + G^{-1}B = I_n + G^{-1}(A - G) = G^{-1}A$, where I_n is the $n \times n$ identity matrix. But $I_n \geq 0$, $G^{-1} > 0$, and $B \geq 0$ with $b_{ii} > 0, \forall i$, so that

$$G^{-1}A > 0. \tag{4}$$

Finally, we assume that a certain number of grid nodes $d \leq n$ are required to satisfy some user-provided voltage drop threshold specifications, captured in the $d \times 1$ vector $V_{\mathrm{th}} \geq 0$. These would typically be nodes at the lower metal layers, where the chip circuitry is connected. Let P be a $d \times n$ matrix consisting of 0 and 1 elements only, specifying (with 1 entry) the nodes that are subject to voltage threshold specification. Note that $P \geq 0$ and P have exactly one 1 entry in every row, otherwise 0s, and that no column of P has more than one 1.

III. Problem Definition

We will introduce the notion of a container for a vector of current waveforms, which will help us express constraints that guarantee grid safety.

Definition 2 (Container): Let $t \in \mathbb{R}$, let $i(t) \in \mathbb{R}^m$ be a bounded function of time, and let $\mathcal{F} \subset \mathbb{R}^m$ be a closed subset of $\mathbb{R}^m$. If $i(t) \in \mathcal{F}$, $\forall t \in \mathbb{R}$, then we say that $\mathcal{F}$ contains $i(t)$, represented by the shorthand $i(t) \subset \mathcal{F}$, and we refer to $\mathcal{F}$ as a container of $i(t)$.

Fig. 1 illustrates the idea of a container for a simple case of two current waveforms. Because $i(t) = [i_1(t)\ i_2(t)]^T \in \mathcal{F}$ for all time instants, we say that $\mathcal{F}$ contains $i(t)$.

Definition 3 (Safe Grid): A grid is said to be safe for a given $i(t)$, defined $\forall t \in \mathbb{R}$, if the resulting $Pv(t) \leq V_{\mathrm{th}}$, $\forall t \in \mathbb{R}$.

To check if a power grid is safe, one would typically be interested in the worst-case voltage drop at some grid node k, at some time point $\tau \in \mathbb{R}$, over a wide range of possible current waveforms. Using the above notation, and given a container $\mathcal{F}$ that contains a wide range of current waveforms that are of interest, we can express this as $\max_{i(t) \subset \mathcal{F}}(v_k(\tau))$. Clearly, because $\mathcal{F}$ is the same irrespective of time and applies at all time points $t \in \mathbb{R}$, then this worst-case voltage drop

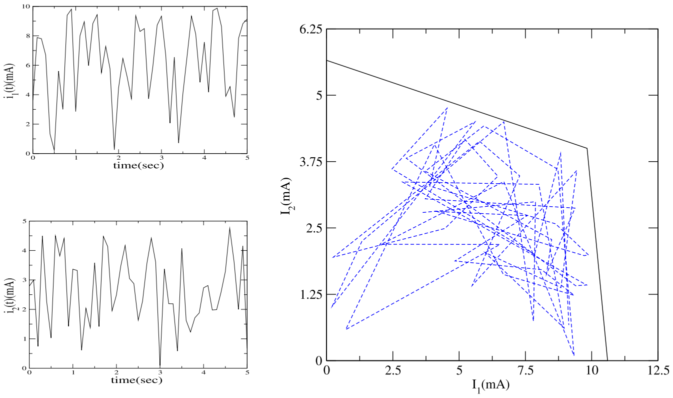

Fig. 1. Example of a container $\mathcal{F}$ for $i_1(t)$ and $i_2(t)$.

must be time-invariant, independent of the chosen time point τ. We now introduce the emax$(\cdot)$ notation, which is used to capture in a single vector of all the separate worst-case voltage drop maximizations, as follows.

Definition 4 (Emax Operator): Let $\mathcal{X}$ be a bounded and closed subset of $\mathbb{R}^n$ and let $x \in \mathbb{R}^n$ be an $n \times 1$ vector. If $y \in \mathbb{R}^n$ is another $n \times 1$ vector such that, for every i, $y_i = \max_{x \in \mathcal{X}} [x_i]$ then we capture this with the shorthand notation

$$y = \operatorname*{emax}_{x \in \mathcal{X}} [x]. \tag{5}$$

For example, if $\mathcal{X} = \{x \in \mathbb{R}^2 : x \geq 0, x_1 + 2x_2 \leq 2\}$, then

$$\operatorname*{emax}_{x \in \mathcal{X}} [x] = \begin{bmatrix} 2 \\ 1 \end{bmatrix}. \tag{6}$$

The name "emax" stands for element-wise maximization and the notation will be used in other forms as well. For example, if $f(x)$ is a vector valued function of x, we may write

$$z = \operatorname*{emax}_{x \in \mathcal{X}} \big[f(x)\big]. \tag{7}$$

Definition 5 (Worst-Case Drop): For a given container $\mathcal{F}$, and for an arbitrary $\tau \in \mathbb{R}$, define

$$v^*(\mathcal{F}) \triangleq \operatorname*{emax}_{i(t) \subset \mathcal{F}} [v(\tau)] = \begin{bmatrix} \max_{i(t) \subset \mathcal{F}} (v_1(\tau)) \\ \max_{i(t) \subset \mathcal{F}} (v_2(\tau)) \\ \vdots \\ \max_{i(t) \subset \mathcal{F}} (v_n(\tau)) \end{bmatrix} \tag{8}$$

to be the $n \times 1$ vector whose every entry is the worst-case voltage drop at the corresponding node under all possible current waveforms $i(t)$ contained in $\mathcal{F}$, with the convention that, if $\mathcal{F} = \phi$, then $\operatorname*{emax}_{i(t) \subset \mathcal{F}} [v(\tau)] = 0$.

The exact expression for $v^*(\mathcal{F})$ was derived in [13] to be

$$v^*(\mathcal{F}) = \sum_{q=0}^{\infty} \operatorname*{emax}_{I \in \mathcal{F}} \left[\left(A^{-1}B\right)^q M'I \right] \tag{9}$$

where $I \in \mathbb{R}^m$ is a vector of artificial variables, with units of current, that is used to carry out the emax$(\cdot)$ operation. One way to check node safety is to compute (9), which would then be compared against the threshold voltages V_{th}. However, this would be prohibitively expensive. Instead, we will use an upper-bound on $v^*(\mathcal{F})$ based on the following.

Definition 6: For any $\mathcal{F} \subset \mathbb{R}^m$, define

$$\bar{v}(\mathcal{F}) \triangleq G^{-1} A \operatorname*{emax}_{I \in \mathcal{F}} \big(M'I\big) \tag{10}$$

where $I \in \mathbb{R}^m$ is a vector of artificial variables, with units of current, that is used to carry out the emax$(\cdot)$ operation, with the convention that $\operatorname{emax}_{I \in \mathcal{F}}(M'I) = 0$, if $\mathcal{F} = \phi$.

Najm [8] and Ferzli *et al.* [13] derived the following upper-bound[1] on $v^*(\mathcal{F})$, for any container $\mathcal{F}$

$$v^*(\mathcal{F}) \leq \bar{v}(\mathcal{F}). \tag{11}$$

Furthermore, Ferzli *et al.* [13] investigated the accuracy of this upper-bound which was found to be quite good (recent tests show a maximum error of 4 mV on a 5 K node grid).

Definition 7 (Safe Container): For a given container $\mathcal{F}$, we say that $\mathcal{F}$ is safe if $P\bar{v}(\mathcal{F}) \leq V_{\text{th}}$.

Thus, we are interested to discover a container $\mathcal{F}$ for which $P\bar{v}(\mathcal{F}) \leq V_{\text{th}}$, so that $Pv^*(\mathcal{F}) \leq V_{\text{th}}$ and the grid is safe. We will see below that a safe container $\mathcal{F}$ can be expressed as a set of constraints on the circuit currents that load the grid, thereby providing a set of linear constraints that are sufficient to guarantee grid safety. We will find, however, that the choice of $\mathcal{F}$ is not unique. Indeed, there is an infinity of possible safe containers. In the following sections, we will first characterize the most desirable safe containers, which we will call maximal, and then develop algorithms to generate specific types of maximal containers for specific objectives.

IV. Maximal Containers

This section contains the bulk of the theoretical contribution of this paper, culminating in the necessary and sufficient conditions given in Theorem 1. It lays the foundation for subsequent sections.

Let $u \in \mathbb{R}^n$ and define the sets $\mathcal{U}$, $\mathcal{F}(u)$, and $\mathcal{S}$ as follows:

$$\mathcal{U} \triangleq \left\{u \in \mathbb{R}^n : u \geq 0, Pu \leq V_{\text{th}}\right\} \tag{12}$$

$$\mathcal{F}(u) \triangleq \left\{I \in \mathbb{R}^m : I \geq 0,\ M'I \leq MGu\right\} \tag{13}$$

$$\mathcal{S} \triangleq \{\mathcal{F}(u) : u \in \mathcal{U}\} \tag{14}$$

and notice that

$$MGu \leq MGu' \Longrightarrow \mathcal{F}(u) \subseteq \mathcal{F}(u'), \quad \forall u, u' \in \mathbb{R}^n. \tag{15}$$

Notice that $\mathcal{U}$ is the set of all safe voltage drop assignments and, as we will see below, $\mathcal{S}$ is the set of all safe containers. We will see that it is enough to consider only containers of the form (13). The first step toward this conclusion is due to the following lemma.

Lemma 3: For any $u \in \mathbb{R}^n_+$, we have $0 \leq \bar{v}(\mathcal{F}(u)) \leq u$.

Proof: For any $u \in \mathbb{R}^n_+$, if $\mathcal{F}(u) = \phi$ then, from Definition 6, $0 = \bar{v}(\mathcal{F}(u)) \leq u$. Otherwise, if $\mathcal{F}(u) \neq \phi$, then Definition 6 provides that $\bar{v}(\mathcal{F}(u)) \geq 0$, due to (4) and the fact that $I \geq 0$, for all $I \in \mathcal{F}(u)$, by definition. Furthermore, from (13), we have $M'I \leq MGu,\ \forall I \in \mathcal{F}(u)$, so that

$$\operatorname*{emax}_{I \in \mathcal{F}(u)} \big(M'I\big) \leq MGu. \tag{16}$$

[1]In [13], the upper bound on the worst case voltage drop is in terms of $i(t)$ and M', as in (10). In [8], the proof is presented in terms of $i_s(t) = Hi(t)$ and M, but it can be easily shown that the upper bound in [8] also implies (10).

Multiplying both sides of (16) with $G^{-1}A \geq 0$, due to (4), we get $\overline{v}(\mathcal{F}(u)) \leq u$, which completes the proof. ■

Building on this and because $i(t) \geq 0$ is already assumed in our grid model, we are only interested in containers that are subsets of $\mathbb{R}_+^m$, so that the next step that establishes the importance of the type of containers in (13) is given by the following necessary and sufficient condition.

Lemma 4: A container $\mathcal{J} \subset \mathbb{R}_+^m$ is safe if and only if it is a member of $\mathcal{S}$ or a subset of a member of $\mathcal{S}$.

Proof: The proof is in two parts.

Proof of the "if Direction": Let $\mathcal{J} \subseteq \mathcal{F}(u)$ for some $u \in \mathcal{U}$, it follows that $\text{emax}_{I\in\mathcal{J}}(M'I) \leq \text{emax}_{I\in\mathcal{F}(u)}(M'I)$, from which $\overline{v}(\mathcal{J}) \leq \overline{v}(\mathcal{F}(u))$, due to (4). Using Lemma 3, we get $\overline{v}(\mathcal{J}) \leq u$ which, due to $u \in \mathcal{U}$ and $P \geq 0$, gives $P\overline{v}(\mathcal{J}) \leq V_{\text{th}}$.

Proof of the "Only if Direction": Let $\mathcal{J} \subset \mathbb{R}_+^m$ with $P\overline{v}(\mathcal{J}) \leq V_{\text{th}}$, and let $u = \overline{v}(\mathcal{J})$, so that $Pu \leq V_{\text{th}}$ and

$$G^{-1}A \operatorname*{emax}_{I\in\mathcal{J}}\big(M'I\big) = u. \tag{17}$$

Because $I \geq 0$ for any $I \in \mathcal{J}$, and due to (4), we have $u \geq 0$, so that $u \in \mathcal{U}$. Multiplying (17) with MG, we get

$$\operatorname*{emax}_{I\in\mathcal{J}}\big(M'I\big) = MGu \tag{18}$$

so that, $\forall I \in \mathcal{J}$, we have $M'I \leq MGu$ which coupled with $I \geq 0$ gives $\mathcal{J} \subseteq \mathcal{F}(u)$. ■

Therefore: 1) $\mathcal{F}(u)$ is safe for any $u \in \mathcal{U}$ and 2) all interesting safe containers $\mathcal{J}$ may be found as either specific $\mathcal{F}(u)$ for some $u \in \mathcal{U}$, or as subsets of such $\mathcal{F}(u)$. Note that, if $\mathcal{J} \subseteq \mathcal{F}(u)$, for some $u \in \mathcal{U}$, with $\mathcal{J} \neq \mathcal{F}(u)$, then clearly $\mathcal{F}(u)$ is a better choice than $\mathcal{J}$. Choosing $\mathcal{J}$ would be unnecessarily limiting, while $\mathcal{F}(u)$ would allow more flexibility in the circuit loading currents. Therefore, it is enough to consider only containers of the form $\mathcal{F}(u)$ with $u \in \mathcal{U}$. This is why the definitions (12)–(14) are important!

Going further, if $\mathcal{F}(u_1) \subseteq \mathcal{F}(u_2)$ with $\mathcal{F}(u_1) \neq \mathcal{F}(u_2)$, then clearly $\mathcal{F}(u_2)$ is a better choice than $\mathcal{F}(u_1)$. Thus, in a sense, the "larger" the container, the better. We capture this with the notion of maximality, as follows.

Definition 8: Let $\mathcal{E}$ be a collection of subsets of $\mathbb{R}^m$ and let $\mathcal{X} \in \mathcal{E}$. We say that $\mathcal{X}$ is maximal in $\mathcal{E}$ if there does not exist another $\mathcal{Y} \in \mathcal{E}$, $\mathcal{Y} \neq \mathcal{X}$, such that $\mathcal{X} \subseteq \mathcal{Y}$.

Maximality is a highly desirable property and so the purpose of the rest of this section is to give necessary and sufficient conditions for a container to be maximal in $\mathcal{S}$. We will see below that the maximality of $\mathcal{F}(u)$ depends on crucial properties of u. Note that first $0 \in \mathcal{U}$ for any $V_{\text{th}} \geq 0$, and $\mathcal{F}(0) = \{0\}$, due to $M' > 0$ combined with $I \geq 0$, $\forall I \in \mathcal{F}(0)$. It follows that $\mathcal{S}$ always contains a nonempty set as a member, so that $\mathcal{F}(u) = \phi$ is never maximal in $\mathcal{S}$—this will be useful as follows.

A. Feasible

Definition 9: For any $u \in \mathbb{R}^n$, u is said to be feasible if $\mathcal{F}(u)$ is not empty, otherwise it is infeasible.

Because $\mathcal{F}(0) = \{0\}$, then $u = 0$ is always feasible. In general, we have the following lemma.

Lemma 5: For any $u \in \mathbb{R}^n$, u is feasible if and only if $MGu \geq 0$.

Proof: To prove the if direction, let $u \in \mathbb{R}^n$ with $MGu \geq 0$, in which case clearly $0 \in \mathcal{F}(u)$, so that $\mathcal{F}(u)$ is not empty and u is feasible. To prove the only if direction, let $u \in \mathbb{R}^n$ be feasible so that $\mathcal{F}(u)$ is not empty, and there exists an $I \in \mathbb{R}^m$ such that $I \geq 0$ and $M'I \leq MGu$. Due to $M' \geq 0$ combined with $I \geq 0$, we have $0 \leq M'I \leq MGu$, so that $MGu \geq 0$. ■

Notice that if $u \in \mathbb{R}^n$ is feasible then multiplying both sides of $MGu \geq 0$ by $G^{-1}A \geq 0$ gives $u \geq 0$, so that, $u \in \mathbb{R}_+^n$. Notice also that, if $V_{\text{th},k} = 0$, then for every $u \in \mathcal{U}$ we have $u_k = 0$. In this case, the only feasible $u \in \mathcal{U}$ is $u = 0$, because otherwise $MG = M(A-B) = I_n - MB$, which leads to $MGu|_k = u_k - MBu|_k = -MBu|_k < 0$.

B. Extremal

Definition 10: For any $u \in \mathcal{U}$, we say that u is extremal in $\mathcal{U}$ if $\exists k \in \{1, \ldots, d\}$ such that $Pu|_k = V_{\text{th},k}$.

Denote by m'_{ij} the (i, j)th element of M' and by c'_j its jth column.

Lemma 6: If $\mathcal{F}(u)$ is maximal in $\mathcal{S}$ then u is feasible and extremal in $\mathcal{U}$.

Proof: We will prove the contrapositive. Let $u \in \mathcal{U}$ be either infeasible or not extremal in $\mathcal{U}$; we will prove that $\mathcal{F}(u)$ is not maximal in $\mathcal{S}$. If u is infeasible then $\mathcal{F}(u) = \phi$, which we already know is not maximal in $\mathcal{S}$. Now consider the case when u is feasible but not extremal in $\mathcal{U}$. In other words, we have $MGu \geq 0$, $u \geq 0$, and $Pu < V_{\text{th}}$, so that $\epsilon \triangleq \min_{\forall i}(V_{\text{th},i} - Pu|_i) > 0$. Let $\mathbb{1}_n$ be the $n \times 1$ vector whose every entry is 1 and let $u' = u + \epsilon \mathbb{1}_n \geq 0$. Because G is irreducibly diagonally dominant with positive diagonal and nonpositive off-diagonal entries, then $G\mathbb{1}_n \geq 0$, with $G\mathbb{1}_n \neq 0$. Let $\gamma \triangleq G(u' - u) = \epsilon G\mathbb{1}_n$, so that $\gamma \geq 0$ with $\gamma \neq 0$, then $M\gamma > 0$ so that $MGu' > MGu$. Furthermore, considering $u' = u + \epsilon\mathbb{1}_n$, we have $Pu' = Pu + \epsilon P\mathbb{1}_n$. Because P has exactly one 1 in each row, it follows that $P\mathbb{1}_n = \mathbb{1}_d$, and $Pu' = Pu + \epsilon\mathbb{1}_d$, from which $Pu' \leq V_{\text{th}}$ due to the definition of ϵ, so that $u' \in \mathcal{U}$. We have so far established that there exists $u' \in \mathcal{U}$ with $MGu < MGu'$, so that $\mathcal{F}(u) \subseteq \mathcal{F}(u')$, due to (15). It only remains to prove that $\mathcal{F}(u) \neq \mathcal{F}(u')$. For some $i \in \{1, \ldots, m\}$, let $j = \text{argmin}_{\forall k}(MGu'|_k / m'_{ki})$, $\delta = (MGu'|_j / m'_{ji}) \geq 0$, $e_i \in \mathbb{R}^m$ be the vector whose ith entry is 1 and all other entries are 0, and $I^{(i)} = \delta e_i \geq 0$. Notice that, for any k, we have

$$M'I^{(i)}\Big|_k = \delta\, M'e_i\big|_k = \delta\, c'_i\big|_k = \delta m'_{ki}. \tag{19}$$

Therefore, $M'I^{(i)}|_j = \delta m'_{ji} = MGu'|_j > MGu|_j$, so that $I^{(i)} \notin \mathcal{F}(u)$. By definition of δ, we have $\delta \leq (MGu'|_k / m'_{ki})$, for every k, meaning $\delta m'_{ki} \leq MGu'|_k$, for every k. Using (19), we then have $M'I^{(i)}|_k = \delta m'_{ki} \leq MGu'|_k$, for every k, which leads to $I^{(i)} \in \mathcal{F}(u')$, and the proof is complete. ■

C. Irreducible

Definition 11: We say that $u \in \mathbb{R}^n$ is reducible if there exists $u' \leq u$, $u' \neq u$, with $\mathcal{F}(u') = \mathcal{F}(u)$; otherwise, u is said to be irreducible.

We will see that irreducibility of u is a crucial property that is required for maximality of $\mathcal{F}(u)$.

Lemma 7: For any feasible $u \in \mathbb{R}^n$ and any $z \in \mathbb{R}^n$ such that $0 \leq MGz \leq MG(u - \overline{v}(\mathcal{F}(u)))$, let $u' = u - z$, it follows that $\mathcal{F}(u') = \mathcal{F}(u)$.

Proof: For any $I \in \mathcal{F}(u')$, we have $I \geq 0$ and $M'I \leq MGu' = MGu - MGz \leq MGu$, because $MGz \geq 0$, so that $I \in \mathcal{F}(u)$. It follows that $\mathcal{F}(u') \subseteq \mathcal{F}(u)$. In addition, for any $I \in \mathcal{F}(u)$, we have $I \geq 0$ and

$$M'I \leq \underset{I \in \mathcal{F}(u)}{\text{emax}}\big(M'I\big) = MG\overline{v}(\mathcal{F}(u)). \tag{20}$$

Notice that for any z with $0 \leq MGz \leq MG(u - \overline{v}(\mathcal{F}(u)))$, we have $MGu' = MGu - MGz \geq MGu - MG(u - \overline{v}(\mathcal{F}(u))) = MG\overline{v}(\mathcal{F}(u))$. Combining this with (20), we get $M'I \leq MGu'$, so that $I \in \mathcal{F}(u')$. Therefore, $\mathcal{F}(u) \subseteq \mathcal{F}(u')$ from which $\mathcal{F}(u') = \mathcal{F}(u)$, and the proof is complete. ■

Lemma 8: For any feasible $u \in \mathbb{R}^n$, let $u' = \overline{v}(\mathcal{F}(u))$, it follows that $\mathcal{F}(u') = \mathcal{F}(u)$.

Proof: Let $z = u - \overline{v}(\mathcal{F}(u))$, so that $MGz = MGu - MG\overline{v}(\mathcal{F}(u)) = MGu - \text{emax}_{I \in \mathcal{F}(u)}(M'I) \geq 0$, the last step due to the definition of $\mathcal{F}(u)$. As a result, z satisfies the conditions of Lemma 7. Let $u' = u - z = \overline{v}(\mathcal{F}(u))$. Then, by Lemma 7, $\mathcal{F}(u') = \mathcal{F}(u)$. ■

Lemma 9: For any $u \in \mathbb{R}^n_+$, u is irreducible if and only if it is feasible and $\overline{v}(\mathcal{F}(u)) = u$.

Proof: The proof is in two parts.

Proof of the if Direction: The proof is by contradiction. Let u be feasible with $\overline{v}(\mathcal{F}(u)) = u$ and suppose that u is reducible so that there exists $u' \leq u$, $u' \neq u$, with $\mathcal{F}(u') = \mathcal{F}(u)$. Notice that $\mathcal{F}(u)$ is not empty, because u is feasible, so that $\mathcal{F}(u')$ is not empty and u' is feasible. Therefore, we get

$$u' - \overline{v}\big(\mathcal{F}\big(u'\big)\big) = u' - \overline{v}(\mathcal{F}(u)) = u' - u + u - \overline{v}(\mathcal{F}(u)).$$

Notice that, because u' is feasible, we have $MGu' \geq 0$ from which $u' \geq 0$, due to (4). Because $\overline{v}(\mathcal{F}(u')) \leq u'$, due to Lemma 3, it follows that $u' - u + u - \overline{v}(\mathcal{F}(u)) \geq 0$, so that $u - \overline{v}(\mathcal{F}(u)) \geq u - u' \geq 0$. But $u - u' \neq 0$, so that $\overline{v}(\mathcal{F}(u)) \neq u$ and we have a contradiction that completes the proof.

Proof of the Only if Direction: We will prove the contrapositive. Let u be either infeasible or $\overline{v}(\mathcal{F}(u)) \neq u$, and we will prove that u is reducible. If u is infeasible then $\mathcal{F}(u) = \phi$ and $u \neq 0$ (recall, $u = 0$ is always feasible), and it is easy to find another infeasible u' with $u' \leq u$ and $u' \neq u$, as follows. Let $u' = (1/2)u$, from which $MGu' = (1/2)MGu \not\geq 0$, because u is infeasible, so that u' is infeasible. Therefore, we have found $u' \leq u$, $u' \neq u$, with $\mathcal{F}(u') = \mathcal{F}(u) = \phi$ which means u is reducible. If u is feasible and $\overline{v}(\mathcal{F}(u)) \neq u$, let $u' = \overline{v}(\mathcal{F}(u)) \in \mathbb{R}^n_+$, due to Lemma 3, which also provides that $\overline{v}(\mathcal{F}(u)) \leq u$, so that $u' \leq u$, $u' \neq u$, with $\mathcal{F}(u') = \mathcal{F}(u)$ due to Lemma 8, and u is reducible. ■

Note, if u is irreducible and extremal in $\mathcal{U}$, then $Pu|_k = V_{\text{th},k}$ for some k, and $P\overline{v}(\mathcal{F}(u))|_k = V_{\text{th},k}$.

Lemma 10: For any $u \in \mathbb{R}^n_+$, u is irreducible if and only if

$$MGu \leq MGu' \Longleftrightarrow \mathcal{F}(u) \subseteq \mathcal{F}\big(u'\big), \quad \forall u' \in \mathbb{R}^n. \tag{21}$$

Proof: The proof is in two parts.

Proof of the if Direction: We give a proof by contradiction. Given (21) and suppose u is reducible, so that it is either infeasible or $\overline{v}(\mathcal{F}(u)) \neq u$. If u is infeasible, then $\mathcal{F}(u) = \phi \subseteq \mathcal{F}(u')$, for any $u' \in \mathbb{R}^n$, so that $MGu \leq MGu'$, for any $u' \in \mathbb{R}^n$, due to (21). But this is impossible, because we can always find a $u' \in \mathbb{R}^n$ that violates $MGu \leq MGu'$, as follows. Let $\mathbb{1}_n$ be the $n \times 1$ vector whose every entry is 1 and let $w = -G^{-1}A\mathbb{1}_n$ so that $MGw = -\mathbb{1}_n < 0$, and let $u' = u + w$ so that $MGu' - MGu = MGw < 0$. Therefore, it must be that u is feasible and $\overline{v}(\mathcal{F}(u)) \neq u$. Let $u' = \overline{v}(\mathcal{F}(u))$, so that $\mathcal{F}(u') = \mathcal{F}(u)$ due to Lemma 8, with $MGu' = MG\overline{v}(\mathcal{F}(u))$. Recall that $MG\overline{v}(\mathcal{F}(u)) = \text{emax}_{I \in \mathcal{F}(u)}(M'I) \leq MGu$, and $MG\overline{v}(\mathcal{F}(u)) \neq MGu$ due to $\overline{v}(\mathcal{F}(u)) \neq u$, so that $MGu' \leq MGu$ and $MGu' \neq MGu$. This means that we have $\mathcal{F}(u) \subseteq \mathcal{F}(u')$ while $MGu \not\leq MGu'$, which contradicts (21), and the proof is complete.

Proof of the Only if Direction: Let u be irreducible, so that u is feasible with $\overline{v}(\mathcal{F}(u)) = u$. Due to (15), it only remains to prove that $\forall u' \in \mathbb{R}^n, \mathcal{F}(u) \subseteq \mathcal{F}(u') \Longrightarrow MGu \leq MGu'$. Notice that $\mathcal{F}(u')$ is nonempty, because $\mathcal{F}(u) \neq \phi$ and $\mathcal{F}(u) \subseteq \mathcal{F}(u')$, from which u' is feasible. Because u and u' are feasible, and using $u = \overline{v}(\mathcal{F}(u))$, notice that

$$\begin{aligned} MGu' - MGu &= MGu' - MG\overline{v}(\mathcal{F}(u)) \\ &= MGu' - \underset{I \in \mathcal{F}(u)}{\text{emax}}\big(M'I\big) \\ &\geq MGu' - \underset{I \in \mathcal{F}(u')}{\text{emax}}\big(M'I\big) \geq 0 \end{aligned}$$

where we used $\text{emax}_{I \in \mathcal{F}(u')}(M'I) \geq \text{emax}_{I \in \mathcal{F}(u)}(M'I)$, because $\mathcal{F}(u) \subseteq \mathcal{F}(u')$. Therefore, $MGu' - MGu \geq 0$, so $MGu \leq MGu'$ and the proof is complete. ■

D. Maximality

As pointed out earlier, we are interested in safe containers that are maximal in $\mathcal{S}$. We now present our main result that gives the necessary and sufficient conditions for maximality.

Theorem 1: $\mathcal{F}(u)$ is maximal in $\mathcal{S}$ if and only if u is irreducible and extremal in $\mathcal{U}$.

Proof: The proof is in two parts.

Proof of the if Direction: We give a proof by contradiction. Let $u \in \mathcal{U}$ be irreducible and extremal in $\mathcal{U}$, but suppose that $\mathcal{F}(u)$ is not maximal in $\mathcal{S}$, so that $\exists u' \in \mathcal{U}$ such that $\mathcal{F}(u) \subseteq \mathcal{F}(u')$, with $\mathcal{F}(u) \neq \mathcal{F}(u')$. Because $\mathcal{F}(u) \neq \mathcal{F}(u')$, then clearly $MGu \neq MGu'$, and using Lemma 10, we have $MGu \leq MGu'$. Let $\delta = MGu' - MGu$, so that $\delta \geq 0$ and $\delta \neq 0$. Because $G^{-1}A > 0$ from (4), then $G^{-1}A\delta = u' - u > 0$. Then $u < u'$ and $Pu < Pu' \leq V_{\text{th}}$, due to $P \geq 0$, P has no row with all zeros, and $u' \in \mathcal{U}$, so that u is not extremal in $\mathcal{U}$, and we have a contradiction that completes the proof.

Proof of the Only if Direction: We give a proof by contradiction. Given that $\mathcal{F}(u)$ is maximal in $\mathcal{S}$, we know from Lemma 6 that u is feasible and extremal in $\mathcal{U}$. Suppose u is reducible, so that $\overline{v}(\mathcal{F}(u)) \neq u$, because we already have that u is feasible. Recall that $0 \leq \overline{v}(\mathcal{F}(u)) \leq u$, from which $P\overline{v}(\mathcal{F}(u)) \leq Pu$, due to $P \geq 0$. Let $u' \triangleq \overline{v}(\mathcal{F}(u)) \neq u$, so that $u' \in \mathcal{U}$ and $MGu' = MG\overline{v}(\mathcal{F}(u)) = \text{emax}_{I \in \mathcal{F}(u)}(M'I)$. Let $\delta = MGu - MGu' = MGu - \text{emax}_{I \in \mathcal{F}(u)}(M'I)$, then we have $\delta \geq 0$ and $\delta \neq 0$ (due to $u' \neq u$). Because $G^{-1}A > 0$, then $G^{-1}A\delta = u - u' > 0$. Consequently, we have $u' < u$, so that $Pu' < Pu \leq V_{\text{th}}$, due to $P \geq 0$, P has no row with all zeros, and $u \in \mathcal{U}$, so that u' is not extremal in $\mathcal{U}$. Therefore, by Lemma 6,

$\mathcal{F}(u')$ is not maximal in $\mathcal{S}$. However, $\mathcal{F}(u) = \mathcal{F}(u')$, due to Lemma 8, so that $\mathcal{F}(u)$ is not maximal in $\mathcal{S}$, a contradiction that completes the proof. ■

This important theoretical result forms the basis for our choice of practical constraints generation algorithms that are guaranteed to give maximal containers, as we will see in the next section. Recall that whenever u is irreducible and extremal in $\mathcal{U}$, then $P\overline{v}(\mathcal{F}(u))|_k = V_{\text{th},k}$, for some k, so that the upper bound on the voltage drop at the kth grid node would be equal to its maximum allowable voltage drop. In other words, a maximal container always causes some node(s) on the grid to experience the maximum allowable voltage drop, at least based on the $\overline{v}(\cdot)$ upper bound.

E. Redundant Constraints

One concern with (13) is that the set of current constraints $M'I \leq MGu$ can be very large for large grids. In this section, we will show that, under certain conditions, some constraints in (13) are redundant, i.e., they can be removed from the system $M'I \leq MGu$ without altering the set $\mathcal{F}(u)$. Hence, these constraints should be discarded, which reduces the number of constraints without affecting the container $\mathcal{F}(u)$. We introduce the following notation to express the familiar matrices G, A, and M in block form:

$$A = \begin{bmatrix} A_1 & A_2 \\ A_2^T & A_3 \end{bmatrix}, M = \begin{bmatrix} M_1 & M_2 \\ M_2^T & M_3 \end{bmatrix}, G = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} \tag{22}$$

where A_1 and M_1 are $m \times m$ matrices; A_2 and M_2 are $m \times (n-m)$ matrices; A_3 and M_3 are $(n-m) \times (n-m)$ matrices; and G_1 and G_2 are $m \times n$ and $(n-m) \times n$ matrices, respectively. Let $w \triangleq MGu = [w^{(1)}\ w^{(2)}]^T$ such that $w^{(1)}$ and $w^{(2)}$ are $m \times 1$ and $(n-m) \times 1$, respectively.

Lemma 11: For any $u \in \mathbb{R}^n$, if $A_3^{-1} G_2 u \geq 0$, then, $\forall x \in \mathbb{R}_+^m$

$$M_1 x \leq w^{(1)} \iff M' x \leq w. \tag{23}$$

Proof: Clearly, if $M'x \leq w$, then $M_1 x \leq w^{(1)}$. To prove that $M_1 x \leq w^{(1)} \implies M'x \leq w$, consider the following:

$$AM = \begin{bmatrix} A_1 & A_2 \\ A_2^T & A_3 \end{bmatrix} \begin{bmatrix} M_1 & M_2 \\ M_2^T & M_3 \end{bmatrix} = \begin{bmatrix} I_m & 0 \\ 0 & I_{n-m} \end{bmatrix} \tag{24}$$

where I_m and I_{n-m} are identity square matrices of sizes $m \times m$ and $(n-m) \times (n-m)$, respectively. Using (24), we have

$$A_2^T M_1 + A_3 M_2^T = 0 \tag{25}$$

$$A_2^T M_2 + A_3 M_3 = I_{n-m} \tag{26}$$

or equivalently

$$A_2^T M_1 = -A_3 M_2^T \tag{27}$$

$$A_2^T M_2 = I_{n-m} - A_3 M_3. \tag{28}$$

Assume that $M_1 x \leq w^{(1)}$, that is

$$M_1 x \leq w^{(1)} = M_1 G_1 u + M_2 G_2 u. \tag{29}$$

Because $A_2^T \leq 0$, multiplying (29) by A_2^T we get

$$A_2^T M_1 x \geq A_2^T M_1 G_1 u + A_2^T M_2 G_2 u. \tag{30}$$

Using (27) and (30), we get

$$-A_3 M_2^T x \geq -A_3 M_2^T G_1 u + A_2^T M_2 G_2 u. \tag{31}$$

Because A_3 is a principal submatrix of A, then A_3 is a nonsingular $\mathcal{M}$-matrix [14], so that A_3^{-1} exists and $A_3^{-1} \geq 0$, due to Lemma 1. Therefore, we can multiply (31) by $-A_3^{-1}$ to get

$$M_2^T x \leq M_2^T G_1 u - A_3^{-1} A_2^T M_2 G_2 u. \tag{32}$$

Now, using (28) and (32), we get

$$\begin{aligned} M_2^T x &\leq M_2^T G_1 u - A_3^{-1}(I_{n-m} - A_3 M_3) G_2 u \\ &= M_2^T G_1 u + M_3 G_2 u - A_3^{-1} G_2 u \\ &\leq M_2^T G_1 u + M_3 G_2 u = w^{(2)} \end{aligned}$$

where we used the fact that $A_3^{-1} G_2 u \geq 0$. Therefore, $M'x \leq w$ and the proof is complete. ■

In other words, if $A_3^{-1} G_2 u \geq 0$, then the system of inequalities $M'I \leq MGu$ can be reduced from n to m, where m is the number of current sources attached to the grid.

Corollary 1: For any $u \in \mathbb{R}^n$, if $Gu \geq 0$, then

$$M_1 x \leq w^{(1)} \iff M' x \leq w. \tag{33}$$

Proof: Clearly, if $M'x \leq w$, then $M_1 x \leq w^{(1)}$. We now prove that $M_1 x \leq w^{(1)} \implies M'x \leq w$. Let $u \in \mathbb{R}^n$ be such that $Gu \geq 0$, so that $G_2 u \geq 0$. Recall that A_3 is a principal submatrix of A, so that A_3^{-1} exists and $A_3^{-1} \geq 0$, it follows that $A_3^{-1} G_2 u \geq 0$. Benefiting from Lemma 11, it follows that $M_1 x \leq w^{(1)} \implies M_2^T x \leq w^{(2)}$, which gives $M'x \leq w$. ■

The above corollary provides a sufficient algebraic condition under which some constraints in (13) are redundant. This will be useful in the following section.

V. Applications

So far, we have shown that a container $\mathcal{F}(u)$ is maximal in $\mathcal{S}$ if and only if u satisfies the conditions of Theorem 1. In this section, we will describe some design objectives and corresponding algorithms for finding specific maximal safe containers $\mathcal{F}(u)$. These algorithms will each be formulated as a maximization of a certain design objective $g(u)$, over all $u \in \mathcal{U}$. Lemma 16 in the Appendix establishes a sufficient condition on $g(\cdot)$ for which the algorithms proposed below will be shown to produce maximal containers.

A. Peak Power Dissipation

An interesting quality metric for a power grid is the peak total power dissipation that it can safely support in the underlying circuit. We refer here to the instantaneous power dissipation, which is conservatively approximated by $V_{\text{dd}} \sum_{j=1}^{m} i_j(t)$. Thus, we are interested in a safe container that is maximal in $\mathcal{S}$ and that allows the highest possible $\sum_{\forall j} I_j$. Generally, one might be interested in the highest weighted sum of the individual currents, i.e., $\sum_{\forall j} q_j I_j$, where $q_j > 0$ is a user-specified weight on the jth current source. This will allow certain areas of the die to support larger power dissipation than other areas. However, in this paper we assume that all current sources have equal weights and, hence, we will be finding the peak total power dissipation that the grid can safely support.

For any $u \in \mathcal{U}$, we define $\sigma(u)$ to be the largest sum of current source values allowed under $\mathcal{F}(u)$

$$\sigma(u) \triangleq \max_{I \in \mathcal{F}(u)} \left(\sum_{j=1}^{m} I_j \right) \tag{34}$$

and we define σ^* to be the largest $\sigma(u)$ achievable over all possible $u \in \mathcal{U}$, that is

$$\sigma^* \triangleq \max_{u \in \mathcal{U}} (\sigma(u)). \tag{35}$$

Let $u_p \in \mathcal{U}$ be such that $\sigma(u_p) = \sigma^*$, and $I^* \in \mathcal{F}(u_p)$ be such that $\sum_{j=1}^{m} I_j^* = \sigma^*$. In general, u_p and I^* may not be unique. Based on (12) and (13), we can express the combined (34) and (35) as the following LP:

$$\begin{aligned} \sigma^* = \text{Max} \sum_{j=1}^{m} I_j & \\ \text{subject to} \quad & \begin{matrix} M'I \le MGu, & Pu \le V_{\text{th}} \\ I \ge 0, & u \ge 0. \end{matrix} \end{aligned} \tag{36}$$

Let $\mathcal{D}$ be the feasible region of the LP (36)

$$\mathcal{D} \triangleq \left\{ (I, u) : I \ge 0, u \ge 0, M'I \le MGu, Pu \le V_{\text{th}} \right\} \tag{37}$$

so that, from the above, we have

$$\sigma^* = \max_{(I,u) \in \mathcal{D}} \left(\sum_{j=1}^{m} I_j \right). \tag{38}$$

Notice that, $(0, 0) \in \mathcal{D}$ so that $\mathcal{D}$ is not empty and all of σ^*, u_p, and I^* are well-defined. Also, for every $(I, u) \in \mathcal{D}$, we have $M'I \le MGu$ and $I \ge 0$ which, because $M' \ge 0$, gives $0 \le M'I \le MGu$ so that u is feasible, due to Lemma 5. Therefore, u_p is feasible and the container $\mathcal{F}(u_p) = \{I \in \mathbb{R}^m : I \ge 0, M'I \le MGu_p\} \ne \phi$ provides the desired current constraints

$$\begin{aligned} i(t) &\ge 0, & \forall t \in \mathbb{R} \\ M'i(t) &\le MGu_p, & \forall t \in \mathbb{R}. \end{aligned}$$

The following lemma establishes the maximality of $\mathcal{F}(u_p)$, based on Theorem 1. Denote by c_j the jth column of M, and notice that $c_j' = c_j$, for every $j \in \{1, 2, \ldots, m\}$. Also, denote by m_{ij} the (i, j)th element of M.

Lemma 12: $\mathcal{F}(u_p)$ is maximal in $\mathcal{S}$.

Proof: Recall that I^* and u_p are well-defined and $(I^*, u_p) \in \mathcal{D}$, so that $M'I^* \le MGu_p$ and $I^* \ge 0$ which, because $M' \ge 0$, gives $0 \le M'I^* \le MGu_p$ and so u_p is feasible due to Lemma 5. We will prove that $\sigma(\cdot)$ satisfies the conditions of Lemma 16, from which $\mathcal{F}(u_p)$ is maximal in $\mathcal{S}$. First, notice that for any $u, u' \in \mathcal{U}$, if $\mathcal{F}(u') = \mathcal{F}(u)$, it follows that $\sigma(u') = \sigma(u)$, due to (34). It remains to prove that for any $u, u' \in \mathcal{U}$, if $MGu' > MGu$, then $\sigma(u') > \sigma(u)$.

For any $u \in \mathcal{U}$, there must exist a vector $I \in \mathcal{F}(u)$ such that $\sigma(u) = \sum_{j=1}^{m} I_j$. Let $\lambda = \min_{\forall i}(MGu'|_i - MGu|_i)/\max_{\forall i,j}(m_{ij})$. Because $MGu' > MGu$ and $M > 0$, it follows that $\lambda > 0$. Also, let $e_1 \in \mathbb{R}^m$ be the vector whose first entry is 1 and all other entries are 0 and let $I' = I + \lambda e_1$. Because $\lambda > 0$, we have $\lambda e_1 \ge 0$, $\lambda e_1 \ne 0$, $I' \ge I \ge 0$, and $I' \ne I$, so that $\sum_{j=1}^{m} I_j' > \sum_{j=1}^{m} I_j$. Furthermore, we have $I' \in \mathcal{F}(u')$, because

$$M'I' = M'I + \lambda M'e_1 = M'I + \lambda c_1' \tag{39}$$

$$= M'I + \frac{\min_{\forall i}(MGu'|_i - MGu|_i)}{\max_{\forall i,j}(m_{ij})} c_1 \tag{40}$$

$$\le MGu + \min_{\forall i}(MGu'|_i - MGu|_i) \mathbb{1}_n \tag{41}$$

$$\le MGu' \tag{42}$$

where in (41) we used $I \in \mathcal{F}(u)$ and $c_1/\max_{\forall i,j}(m_{ij}) \le \mathbb{1}_n$. Therefore, we have $\sigma(u') \ge \sum_{j=1}^{m} I_j' > \sum_{j=1}^{m} I_j = \sigma(u)$, so that $\sigma(\cdot)$ satisfies the conditions of Lemma 16 and $\mathcal{F}(u_p)$ is maximal in $\mathcal{S}$. ■

Maximality is an all-important property, and is guaranteed by the above lemma. However, we also want to ensure good computational performance, and the next lemma will help achieve that. In fact, the importance of the following lemma is twofold. First, it simplifies the LP (36) into (48) achieving a huge speedup, as we will see in Section VI. Second, it shows that, after solving for u_p, the resulting $\mathcal{F}(u_p)$ can be represented using only m rows of $M'I \le MGu_p$.

Lemma 13: Let $u^* = G^{-1}HI^*$, then $u^* \in \mathcal{U}$ and $\sigma(u^*) = \sigma^*$.

Proof: Recall that $M' = MH \ge 0$ and $I^* \ge 0$, so that $M'I^* \ge 0$. Moreover, because $I^* \in \mathcal{F}(u_p)$, we have

$$0 \le M'I^* = MHI^* \le MGu_p. \tag{43}$$

Multiplying (43) with $G^{-1}A \ge 0$, from (4), we get

$$0 \le G^{-1}HI^* \le u_p. \tag{44}$$

Therefore, we have $0 \le u^* = G^{-1}HI^* \le u_p$, so that $Pu^* \le Pu_p \le V_{\text{th}}$, the final step is due to $u_p \in \mathcal{U}$. It follows that $u^* \in \mathcal{U}$. Moreover, we have that $MGu^* = MHI^* = M'I^*$, from which $I^* \in \mathcal{F}(u^*)$, so that $\sigma(u^*) = \sigma^*$, due to (34), and the proof is complete. ■

Recall that u_p is defined to be any vector $u \in \mathcal{U}$ such that $\sigma(u) = \sigma^*$. Therefore, using Lemma 13, we can let $u_p = G^{-1}HI^*$. Define the set $\mathcal{D}'$ as follows:

$$\mathcal{D}' \triangleq \left\{ (I, u) : I \ge 0, u \ge 0, Pu \le V_{\text{th}}, u = G^{-1}HI \right\}. \tag{45}$$

Notice that, for any $(I, u) \in \mathcal{D}'$, we have $u = G^{-1}HI$, so that $MGu = MHI = M'I$ which, combined with $I \ge 0$, $u \ge 0$, and $Pu \le V_{\text{th}}$, gives $(I, u) \in \mathcal{D}$. Therefore, we have $\mathcal{D}' \subseteq \mathcal{D}$. Also, because $(I^*, u_p) \in \mathcal{D}'$, then $\sigma^* = \max_{(I,u) \in \mathcal{D}'} (\sum_{j=1}^{m} I_j)$, which can be found using the LP

$$\begin{aligned} \sigma^* = \text{Max} \sum_{j=1}^{m} I_j & \\ \text{subject to} \quad & \begin{matrix} u = G^{-1}HI, & Pu \le V_{\text{th}} \\ I \ge 0, & u \ge 0. \end{matrix} \end{aligned} \tag{46}$$

Recall that $H = [I_m \; 0]^T$, so that for every $(I, u) \in \mathcal{D}'$, we have

$$Gu = \begin{bmatrix} G_1 u \\ G_2 u \end{bmatrix} = HI = \begin{bmatrix} I_m \\ 0 \end{bmatrix} I = \begin{bmatrix} I \\ 0 \end{bmatrix} \tag{47}$$

from which $G_1 u = I$ and $G_2 u = 0$. Using (47), we can rewrite (46) as

$$\begin{aligned} \sigma^* = \text{Max}\, \mathbb{1}_m^T G_1 u & \\ \text{subject to} \quad & \begin{matrix} G_1 u \ge 0, & G_2 u = 0 \\ Pu \le V_{\text{th}}, & u \ge 0 \end{matrix} \end{aligned} \tag{48}$$

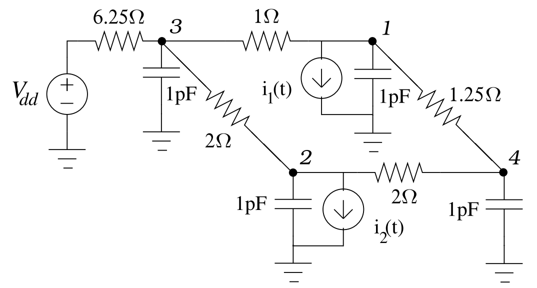

Fig. 2. Example of a power grid with four nodes, two current sources, and $V_{\text{th}} = [110\ 100\ 95\ 105]^T$ (units of mV).

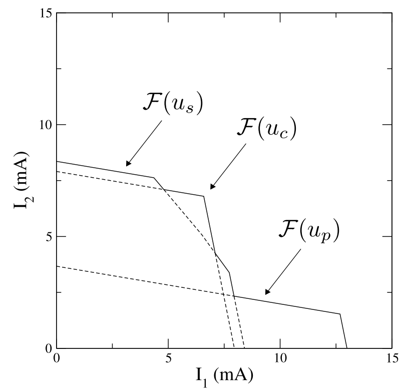

Fig. 3. Example of $\mathcal{F}(u_p)$, $\mathcal{F}(u_s)$, and $\mathcal{F}(u_c)$.

where $\mathbb{1}_m$ is an $m \times 1$ vector whose every entry is 1. The LP in (48) has a remarkable simplification over (36) for two reasons: 1) it has only n variables and $2n$ constraints compared to $n+m$ variables and $2n+m$ constraints and 2) $I_n - MB = I_n - M(A-G) = MG$ which means that MG is a dense matrix, because I_n and B are diagonal matrices and M is a dense matrix, so that the constraints of (36) are dense whereas the constraints of (48) are sparse.

Furthermore, because $u_p = G^{-1}HI^*$, then $Gu_p = [I^*\ 0]^T \geq 0$. Hence, using the corollary to Lemma 11 and for $w = MGu_p$, it follows that $M_1 I \leq w^{(1)} \iff M'I \leq w$, i.e., $r_j' I \leq r_j G u_p$ is redundant, $\forall j \in \{m+1, \ldots, n\}$, where r_j' denotes the jth row of M'. This being said, the container $\mathcal{F}(u_p)$ can be expressed as $\mathcal{F}(u_p) = \{I \in \mathbb{R}^m : I \geq 0, r_j' I \leq r_j G u_p,\ \forall j \in \{1, \ldots, m\}\}$ which provides the desired current constraints

$$\begin{aligned} i(t) &\geq 0, && \forall t \in \mathbb{R} \\ r_j' i(t) &\leq r_j G u_p, \quad \forall j \in \{1, \ldots, m\}, && \forall t \in \mathbb{R}. \end{aligned}$$

As an example, the LP (48) is run on the small grid in Fig. 2 and the resulting container is shown in Fig. 3, where $u_p = [89\ 100\ 95\ 98]^T$ (units of mV). Notice that this method, in order to allow the maximum peak power, may generate a container that is skewed in a way that imposes a tight constraint on current in certain locations of the die [such as at $i_2(t)$] while allowing larger current in other locations [such as at $i_1(t)$]. Other approaches are possible to avoid this skew and even out the current budgets, as we will see next.

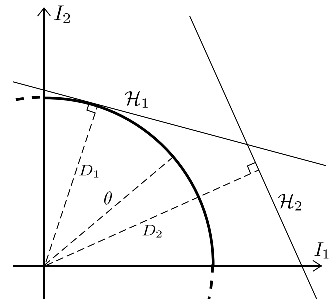

Fig. 4. Illustration of perpendicular distances to hyperplanes.

B. Uniform Current Distribution

The design team may be interested in a grid that safely supports a uniform current distribution across the die, so as to allow a placement that provides a uniform temperature distribution. We can generate constraints that allow that objective by searching for a safe maximal container $\mathcal{F}(u)$ that contains the hypersphere in current space that has the largest volume, or the largest radius θ. In other words, this method aims to "raise the minimum" and avoid the skew indicated above. We will develop a method (54) which, when applied to the simple grid in Fig. 2, generates the container $\mathcal{F}(u_s)$ as shown in Fig. 3, where $u_s = [83\ 91\ 95\ 92]^T$ (units of mV).

Let $S(\theta) \subset \mathbb{R}^m$ denote the hypersphere with radius θ, centered at the origin and let $S^+(\theta) = S(\theta) \cap \mathbb{R}_+^m$ be the part of that hypersphere that is in the first quadrant of $\mathbb{R}^m$. Denote by r_i the ith row of M. For any $u \in \mathcal{U}$, define $\mathcal{H}_i = \{I \in \mathbb{R}^m : I \geq 0, r_i' I = r_i G u\}$ to be the hyperplane that constitutes the ith outer boundary of $\mathcal{F}(u)$, as in the 2-D example in Fig. 4. Define D_i to be the distance from the origin to $\mathcal{H}_i$ which, according to [15], can be expressed as

$$D_i = \frac{|r_i G u|}{d_i}$$

where $d_i = \sqrt{\sum_{j=1}^m m_{ij}^2} > 0$. As we are interested in a nonempty $\mathcal{F}(u)$, we will enforce that $\theta \geq 0$ and u is feasible, i.e., $r_i G u \geq 0$, $\forall i$, so that

$$D_i = \frac{r_i G u}{d_i}. \tag{49}$$

In order to have $S^+(\theta) \subseteq \mathcal{F}(u)$, we will require that

$$\theta \leq D_i, \quad \forall i \in \{1, \ldots, n\} \tag{50}$$

which can be expressed compactly as

$$\theta d \leq MGu \tag{51}$$

where $d = [d_1 \cdots d_n]^T$. For any $u \in \mathcal{U}$, we define $\rho(u)$ to be the largest $\theta \geq 0$ for which $S^+(\theta) \subseteq \mathcal{F}(u)$, or equivalently, for which (51) is satisfied, so that

$$\rho(u) \stackrel{\triangle}{=} \max_{S^+(\theta) \subseteq \mathcal{F}(u)} (\theta) = \max_{0 \leq \theta d \leq MGu} (\theta) \tag{52}$$

and we define ρ^* to be the largest $\rho(u)$ achievable over all possible $u \in \mathcal{U}$, that is

$$\rho^* \stackrel{\triangle}{=} \max_{u \in \mathcal{U}} (\rho(u)). \tag{53}$$

Let u_s be a vector at which the above maximization attains its maximum. In other words, $u_s \in \mathcal{U}$ is such that $\rho(u_s) = \rho^*$ and $S^+(\rho^*) \subseteq \mathcal{F}(u_s)$. In general, u_s may not be unique. We can express the combined (52) and (53) as the following LP:

$$\rho^* = \text{Max}\theta \quad \text{subject to} \quad \begin{array}{ll} \theta d \leq MGu, & Pu \leq V_{\text{th}} \\ \theta \geq 0, & u \geq 0. \end{array} \tag{54}$$

Let $\mathcal{R}$ be the feasible region of the LP (54)

$$\mathcal{R} \triangleq \{(\theta, u) : \theta \geq 0, u \geq 0, \theta d \leq MGu, Pu \leq V_{\text{th}}\} \tag{55}$$

so that, from the above, we have

$$\rho^* = \max_{(\theta,u)\in\mathcal{R}} (\theta). \tag{56}$$

Notice that, $(0, 0) \in \mathcal{R}$ so that $\mathcal{R}$ is not empty and ρ^* and u_s are well-defined. Also, for every $(\theta, u) \in \mathcal{R}$, we have $\theta d \leq MGu$ and $\theta \geq 0$. Because $d \geq 0$, it follows that $0 \leq \theta d \leq MGu$ so that u is feasible, due to Lemma 5. Therefore, u_s is feasible and the container $\mathcal{F}(u_s) = \{I \in \mathbb{R}^m : I \geq 0, M'I \leq MGu_s\} \neq \phi$ provides the desired current constraints

$$\begin{array}{ll} i(t) \geq 0, & \forall t \in \mathbb{R} \\ M'i(t) \leq MGu_s, & \forall t \in \mathbb{R}. \end{array}$$

Lemma 17 in the Appendix, based on Theorem 1, establishes the maximality of $\mathcal{F}(u_s)$.

Another way to write the LP (54) is as follows:

$$\rho^* = \text{Max}\theta \quad \text{subject to} \quad \begin{array}{ll} \theta d \leq w, & Aw = Gu \\ Pu \leq V_{\text{th}}, & \theta, u \geq 0. \end{array} \tag{57}$$

Although the LP (57) has larger number of variables compared to the LP (54), a huge runtime advantage is attained by solving (57) for two reasons: 1) computing all columns of $M = A^{-1}$ is not required in (57), where only m columns of M are sufficient to compute d and 2) notice that $I_n - MB = I_n - M(A - G) = MG$ which means that MG is a dense matrix, because I_n and B are diagonal matrices and M is a dense matrix, so that the constraints of (54) are dense whereas the constraints of (57) are sparse. This being said, we will use the LP (57) to find $\mathcal{F}(u_s)$.

C. Combined Objective

Thus far, we have presented two algorithms for current constraints generation. The first algorithm aims to maximize the peak power dissipation that the grid can safely support in the underlying circuit; however, it generates a skewed container in a way that imposes a tight constraint on the currents in certain locations of the die. The second algorithm aims to uniformly distribute power budgets across the circuit by raising the minimum; but this approach does not necessarily allow for a large peak total power dissipation. One may be interested in a middle scenario; a container that is maximal in $\mathcal{S}$, maximizes the peak power dissipation that the grid can safely support, and supports a uniform current distribution across the die. In this section, we will develop a constraints generation algorithm, essentially a combination of (38) and (56), that allows this type of design objective. The algorithm will generate a container such as the one shown in Fig. 3, where $u_c = [83.6\ 91.4\ 95\ 92.8]^T$ (units of mV).

Recall that (34) maximizes the sum of the m current sources attached to the grid, whereas (52) maximizes the current radius for which the part of the hypersphere in the first quadrant is contained in $\mathcal{F}(u)$. Therefore, there is a clear disproportionality between the dimensions of both objective functions which motivates the following. For any $u \in \mathcal{U}$, we define $\xi(u)$ to be the largest value of the following combined objective allowed under $\mathcal{F}(u)$:

$$\xi(u) \triangleq \max_{\substack{I\in\mathcal{F}(u) \\ S^+(\theta)\subseteq\mathcal{F}(u)}} \left[\left(\sum_{j=1}^{m} I_j \right) + m\theta \right] \tag{58}$$

$$= \max_{\substack{M'I\leq MGu \\ \theta d\leq MGu \\ I,\theta\geq 0}} \left[\left(\sum_{j=1}^{m} I_j \right) + m\theta \right] \tag{59}$$

and we define ξ^* to be the largest $\xi(u)$ achievable under all possible $u \in \mathcal{U}$, so that

$$\xi^* \triangleq \max_{u\in\mathcal{U}} (\xi(u)). \tag{60}$$

Let u_c be a vector at which the above maximization attains its maximum. In other words, $u_c \in \mathcal{U}$ is such that $\xi(u_c) = \xi^*$. Also, let ζ and ω be such that $(\sum_{j=1}^{m} \zeta_j) + m\omega = \xi^*$, where $\zeta \in \mathcal{F}(u_c)$ and $0 \leq \omega d \leq MGu_c$. In general, u_c, ζ, and ω may not be unique. Based on (12) and (13), we can express the combined (58) and (60) as the following LP:

$$\xi(u) = \text{Max} \quad \left(\sum_{j=1}^{m} I_j \right) + m\theta \quad \text{subject to} \quad \begin{array}{ll} M'I \leq MGu, & \theta d \leq MGu \\ Pu \leq V_{\text{th}}, & I, \theta, u \geq 0. \end{array} \tag{61}$$

Let $\mathcal{C}$ be the feasible region of the LP (61)

$$\mathcal{C} \triangleq \left\{ \begin{array}{lll} (I, \theta, u) : & 0 \leq \theta d \leq MGu, & M'I \leq MGu, \\ & I \geq 0, u \geq 0, & Pu \leq V_{\text{th}} \end{array} \right\}$$

so that, from the above, we have

$$\xi^* = \max_{(I,\theta,u)\in\mathcal{C}} \left[\left(\sum_{j=1}^{m} I_j \right) + m\theta \right]. \tag{62}$$

Notice that, $(0, 0, 0) \in \mathcal{C}$ so that $\mathcal{C}$ is not empty, and all of ξ^*, u_c, ζ, and ω are well-defined. Also, for every $(I, \theta, u) \in \mathcal{C}$, we have $M'I \leq MGu$ and $I \geq 0$ which, because $M' \geq 0$, gives $0 \leq M'I \leq MGu$ so that u is feasible, due to Lemma 5. Therefore, u_c is feasible and the container $\mathcal{F}(u_c) = \{I \in \mathbb{R}^m : I \geq 0, M'I \leq MGu_c\}$ provides the desired current constraints

$$\begin{array}{ll} i(t) \geq 0, & \forall t \in \mathbb{R} \\ M'i(t) \leq MGu_c, & \forall t \in \mathbb{R}. \end{array}$$

Lemma 18 in the Appendix establishes the maximality of $\mathcal{F}(u_c)$, based on Theorem 1.

TABLE I
COMPARISON OF THE THREE APPROACHES

Power Grid	**Peak Power**		**Uniform Current Distribution**		**Combined Objective**	
Name	$P(u_p)$ in mW	$\rho(u_p)$ in μA	$P(u_s)$ in mW	$\rho(u_s)$ in μA	$P(u_c)$ in mW	$\rho(u_c)$ in μA
G1	9.83	0.80	5.09	1.80	8.90	1.77
G2	22.06	0.45	11.07	1.44	21.57	1.42
G3	40.69	0.98	22.70	2.11	37.20	2.00
G4	59.77	0.98	35.23	2.03	54.87	1.98
G5	85.23	0.61	50.10	1.61	83.46	1.60
G6	197.57	0.89	116.19	1.94	184.22	1.92
G7	344.34	0.87	205.21	1.84	326.79	1.83
G8	467.86	0.46	269.76	1.48	462.35	1.47
G9	534.13	0.98	319.34	2.03	484.34	2.03

TABLE II
RUNTIME BREAKDOWN OF THE THREE APPROACHES

Power Grid				**Peak Power**		**Uniform Current Distribution**		**Combined Objective**	
Name	Nodes	Current Sources	SPAI Time	LP CPU Time	Total Time	LP CPU Time	Total Time	LP CPU Time	Total Time
G1	50,444	3,192	36.96 sec	4.55 sec	41.51 sec	6.41 sec	43.37 sec	28.93 sec	1.10 min
G2	113,304	7,140	2.98 min	10.27 sec	3.15 min	19.67 sec	3.30 min	1.65 min	4.63 min
G3	200,828	12,656	9.03 min	28.43 sec	9.50 min	39.71 sec	9.61 min	4.73 min	13.76 min
G4	312,232	19,460	21.53 min	56.63 sec	22.47 min	1.14 min	22.67 min	8.24 min	29.77 min
G5	449,189	28,056	44.27 min	2.06 min	46.33 min	2.20 min	46.47 min	14.69 min	58.96 min
G6	1,006,625	63,001	3.58 hrs	9.12 min	3.73 hrs	4.96 min	3.66 hrs	55.55 min	4.50 hrs
G7	1,791,294	111,890	13.68 hrs	21.30 min	14.03 hrs	15.40 min	13.93 hrs	2.34 hrs	16.02 hrs
G8	2,432,119	151,710	24.93 hrs	32.48 min	25.47 hrs	20.79 min	25.27 hrs	5.49 hrs	30.42 hrs
G9	2,795,602	174,306	33.00 hrs	45.22 min	33.75 hrs	26.97 min	33.45 hrs	5.52 hrs	38.52 hrs

As we have transformed (54) into (57), one can write the LP (61) as follows:

$$\xi^* = \text{Max} \quad \left(\sum_{j=1}^{m} I_j\right) + m\theta$$
$$\text{subject to} \quad \begin{array}{ll} y \leq w, & \theta d \leq w \\ Ay = HI, & Aw = Gu \\ Pu \leq V_{\text{th}}, & I, \theta, u \geq 0. \end{array} \tag{63}$$

The LP (63) has more variables and constraints; however, it has more sparse constraints and it only requires m columns of $M = A^{-1}$.

VI. RESULTS

The above three algorithms (48), (57), and (63) have been implemented using C++. Algorithms (57) and (63) require the computation of m columns of the inverse of A (to compute d), where m is the number of current sources attached to the grid, which was performed using the SPAI technique, as was done in [4]. The maximizations were performed using the Mosek optimization package [16]. We conducted tests on a set of power grids with a 1.1 V supply voltage that were generated based on user specifications, including grid dimensions, metal layers, pitch and width per layer, and $C4$ and current source distributions, consistent with 65 nm technology. All results were obtained using a 3.4 GHz Linux machine with 32 GB of RAM.

The number of variables in (48) is n, the number of variables in (57) is $2n+1$, and the number of variables in (63) is $3n+m+1$, where n is the total number of nodes. The CPU times for solving (48), (57), and (63) are given in columns 5, 7, and 9 of Table II, respectively. Note that these CPU times do not include the time for computing the approximate inverse, which is reported separately in column 4. The total CPU time for solving the three approaches is reported in columns 6, 8, and 10 of Table II, respectively. To study runtime efficiency of (48) and (57) compared to the algorithms in [9], the algorithms in [9] were implemented on the machine described above. On average, (48) achieves 43× speedup and (57) achieves 48× speedup, compared to the corresponding algorithms in [9]. For example, on a 310 k nodes grid, the peak power dissipation algorithm in [9] took 13.35 h in total, whereas (48) took 22.47 min, and the uniform current distribution algorithm in [9] took 15.34 h in total, whereas (57) took 22.67 min. The only source of error is the SPAI of A^{-1}, which is controlled by enforcing an error tolerance of 10^{-4} between every entry of the exact inverse and the corresponding entry of the approximate inverse. This error will only affect the computation of d in (57) and (63).

In Table I, we present the results of the three LPs in columns 2–7. Denote by $P(u) \triangleq V_{\text{dd}} \times \sigma(u)$ the peak power dissipation allowed under $\mathcal{F}(u)$. To study the difference between the containers generated using (48), (57), and (63), we used two methods. First, we computed the peak power dissipation achievable under all containers, which are $P(u_p)$, $P(u_s)$, and $P(u_c)$, and the largest current radius for which the part of the hypersphere in the first quadrant is contained in all containers, which are $\rho(u_p)$, $\rho(u_s)$, and $\rho(u_c)$. For instance, on a 449 189 node grid, the peak power dissipation achievable under $\mathcal{F}(u_p)$, $\mathcal{F}(u_s)$, and $\mathcal{F}(u_c)$ is 85.23 mW, 50.1 mW, and 83.46 mW, respectively, and the largest current

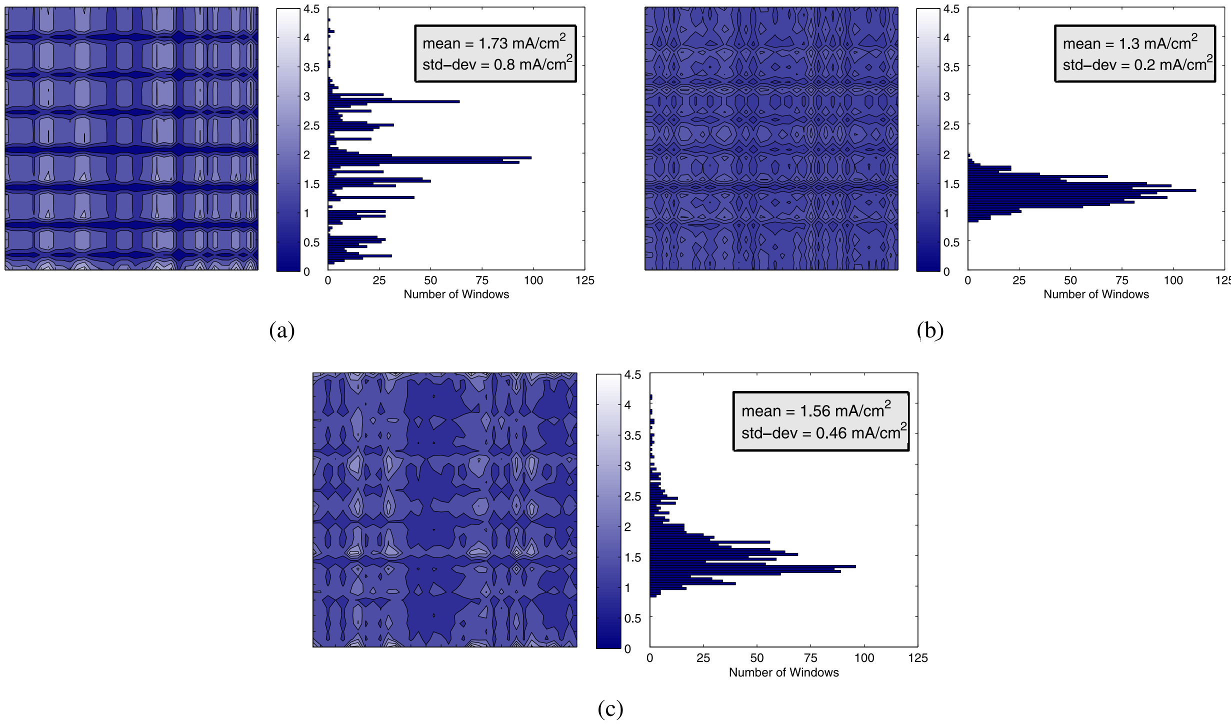

Fig. 5. Contour plots for peak power density across the layout and the corresponding histograms. The color bar units are mA/cm^2. Using (a) $\mathcal{F}(u_p)$, (b) $\mathcal{F}(u_s)$, and (c) $\mathcal{F}(u_c)$.

radius for which the part of the hypersphere in the first quadrant is contained in $\mathcal{F}(u_p)$, $\mathcal{F}(u_s)$, and $\mathcal{F}(u_c)$ is 0.61 μA, 1.61 μA, and 1.60 μA, respectively. The results show that $P(u_s) \ll P(u_p)$ and $\rho(u_p) \ll \rho(u_s)$ on all grids. In fact, the peak power dissipation achievable under $\mathcal{F}(u_s)$ is at most 59% of that achievable under $\mathcal{F}(u_p)$. Also, the largest current radius for which the part of the hypersphere in the first quadrant is contained in $\mathcal{F}(u_p)$ is at most 50% of that contained in $\mathcal{F}(u_s)$. Thus, both $\mathcal{F}(u_p)$ and $\mathcal{F}(u_s)$ provide a distinct tradeoff for the chip design team. Moreover, the results show that $P(u_c) \approx P(u_p)$ and $\rho(u_c) \approx \rho(u_s)$. In fact, $P(u_c)$ is at most 9% less than $P(u_p)$ and $\rho(u_c)$ is at most 7% less than $\rho(u_s)$. Therefore, the combined objective approach gives the best features of the peak power dissipation and the uniform current distribution approaches.

Another way to compare the three approaches (48), (57), and (63), is to look at the power density, i.e., the power dissipation per unit area of the die, allowed by the three resulting containers. To assess this, we maximize the allowed power (current) within a small window of the die surface, and we do this for every position of that window across the die. We divide the die area into $\kappa \times \kappa$ of these windows and compute the peak power density inside each, as allowed by $\mathcal{F}(u_p)$, $\mathcal{F}(u_s)$, or $\mathcal{F}(u_c)$. In Fig. 5(a)–(c), we present contour plots for $\kappa = 35$ for the peak power densities under $\mathcal{F}(u_p)$, $\mathcal{F}(u_s)$, and $\mathcal{F}(u_c)$, respectively, on a 50 k node grid. Note that the current constraints based on $\mathcal{F}(u_p)$ not only allow higher current densities at certain spots but also include some spots with very small and restricted current density budgets. This large spread in power densities can lead to thermal hotspots. This may be avoided by using $\mathcal{F}(u_s)$ which, as expected and as seen in the figure, provides a uniform distribution of power densities across the die area compared to $\mathcal{F}(u_p)$, which is reflected in a smaller standard deviation. Of course, $\mathcal{F}(u_p)$ supports a larger overall peak power dissipation than $\mathcal{F}(u_s)$, which is reflected in a larger mean. The current constraints based on $\mathcal{F}(u_c)$ provide a power density distribution over a smaller range compared to $\mathcal{F}(u_p)$ and allows for larger power dissipation compared to $\mathcal{F}(u_s)$. Clearly, $\mathcal{F}(u_c)$ is superior to $\mathcal{F}(u_p)$ and $\mathcal{F}(u_s)$ providing the best features in those containers.

VII. Conclusion

Early power grid verification is a key step in modern chip design. Traditionally, it has been performed either by simulation or by vectorless verification, both of which have serious shortcomings. We propose a novel method to solve the inverse problem of vectorless verification, by generating circuit current constraints that ensure power grid safety. We develop some key theoretical results to allow the generation of constraints that correspond to maximal current spaces. We then apply these results to provide two constraints generation algorithms that target key quality metrics of the grid: 1) the maximum power dissipation the grid can safely support and 2) the uniformity of the power spread across the die. Finally, we provide a constraints generation algorithm that targets a combination of those quality metrics—this algorithm gives the best features of those two algorithms.

Appendix

Lemma 14: For any feasible $u \in \mathbb{R}^n_+$, let $u' = \overline{v}(\mathcal{F}(u))$, it follows that u' is irreducible.

Proof: For any $u \in \mathbb{R}^n_+$, let $u' = \overline{v}(\mathcal{F}(u))$. Notice that $MGu' = MG\overline{v}(\mathcal{F}(u)) = \text{emax}_{I \in \mathcal{F}(u)}(M'I) \geq 0$ due to $M' \geq 0$

and $I \geq 0$ for any $I \in \mathcal{F}(u)$, so that u' is feasible, due to Lemma 5. Because $u' = \overline{v}(\mathcal{F}(u))$, it follows from Lemma 8 that $\mathcal{F}(u') = \mathcal{F}(u)$, from which $\overline{v}(\mathcal{F}(u')) = \overline{v}(\mathcal{F}(u))$. With this, notice that $u' - \overline{v}(\mathcal{F}(u')) = u' - \overline{v}(\mathcal{F}(u)) = 0$, from which $\overline{v}(\mathcal{F}(u')) = u'$. Using Lemma 9, it follows that u' is irreducible, and the proof is complete. ■

The following lemma provides a necessary algebraic condition for u to be irreducible, which becomes a necessary and sufficient condition in the case $m = n$, i.e., when every grid node is connected to a current source.

Lemma 15: For any $u \in \mathbb{R}^n$, let $w \triangleq MGu$, then we have the following.

1) If u is irreducible then

$$\frac{w_i}{m_{ii}} \leq \frac{w_j}{m_{ji}}, \quad \forall i, j \in \{1, \ldots, n\}. \tag{64}$$

2) In the case $m = n$, if (64) holds then u is irreducible.

Proof: The proof is in two parts.

Proof of 1: Let $u \in \mathbb{R}^n_+$ be irreducible, so that $\overline{v}(\mathcal{F}(u)) = G^{-1}A\,\mathrm{emax}_{I\in\mathcal{F}(u)}(M'I) = u$. Multiplying both sides by MG, we get $\mathrm{emax}_{I\in\mathcal{F}(u)}(M'I) = w$. Then, for every $i \in \{1, \ldots, n\}$, there exists a $y^{(i)} \geq 0$, $M'y^{(i)} \leq w$, and $M'y^{(i)}\big|_i = w_i$, from which

$$r'_j y^{(i)} \leq w_j, \quad \forall j \in \{1, \ldots, n\} \tag{65}$$
$$r'_i y^{(i)} = w_i. \tag{66}$$

After expanding the dot products, we get

$$w_j \geq \sum_{k=1}^{m} m_{jk} y_k^{(i)}, \quad \forall j \in \{1, \ldots, n\} \tag{67}$$
$$w_i = \sum_{k=1}^{m} m_{ik} y_k^{(i)}. \tag{68}$$

For every j, multiply (67) by m_{ii} and (68) by m_{ji}, then subtract the second equation from the first, to get

$$m_{ii}w_j - m_{ji}w_i \geq \sum_{k=1}^{m} \big(m_{ii}m_{jk} - m_{ji}m_{ik}\big) y_k^{(i)} \tag{69}$$

M being the inverse of an $\mathcal{M}$-matrix, the path product condition holds [17], so $m_{ii}m_{jk} \geq m_{ji}m_{ik}$, $\forall i, j, k$, and so the right-hand side of (69) is non-negative. In turn, this means that the left-hand side is non-negative, which leads directly to (64) and completes the proof.

Proof of 2: For any $i \in \{1, 2, \ldots, n\}$, let $e_i \in \mathbb{R}^n$ be the vector whose ith entry is 1 and all other entries are 0, and let $x^{(i)} = (w_i/m_{ii})e_i \geq 0$. Then

$$Mx^{(i)} = \frac{w_i}{m_{ii}} Me_i = \begin{bmatrix} \frac{w_i}{m_{ii}} m_{1i} & \cdots & \frac{w_i}{m_{ii}} m_{ni} \end{bmatrix}^T \leq w \tag{70}$$

where the final inequality is due to (64). Hence, $x^{(i)} \in \mathcal{F}(u)$. With $Mx^{(i)}\big|_i = w_i m_{ii}/m_{ii} = w_i$ due to (70), it follows that $\mathrm{emax}_{I\in\mathcal{F}(u)}(MI) = w = MGu$, so that $\overline{v}(\mathcal{F}(u)) = G^{-1}A\,\mathrm{emax}_{I\in\mathcal{F}(u)}(MI) = u$ and u is irreducible. ■

Lemma 16: Given a real-valued function $g(\cdot) : \mathbb{R}^n \to \mathbb{R}$ such that, for any $u, u' \in \mathcal{U}$, we have the following.

1) $g(u') = g(u)$ if $\mathcal{F}(u') = \mathcal{F}(u)$.
2) $g(u') > g(u)$ if $MGu' > MGu$.

Furthermore, let

$$g^* \triangleq \max_{u\in\mathcal{U}} \big[g(u)\big] \tag{71}$$

and let $u^* \in \mathcal{U}$ be feasible with $g(u^*) = g^*$. It follows that $\mathcal{F}(u^*)$ is maximal in $\mathcal{S}$.

Proof: We will prove that u^* is irreducible and extremal in $\mathcal{U}$, so that $\mathcal{F}(u^*)$ is maximal in $\mathcal{S}$, due to Theorem 1. The proof is in two parts.

First, we will prove that u^* is extremal in $\mathcal{U}$; the proof will be by contradiction. Let $u \in \mathcal{U}$ be feasible with $g(u) = g^*$ and suppose that u is not extremal in $\mathcal{U}$, so that $u \geq 0$ and $Pu < V_{\text{th}}$. Let $\epsilon \triangleq \min_{\forall k}(V_{\text{th},k} - Pu|_k) > 0$, let $\mathbb{1}_n$ be the $n \times 1$ vector whose every entry is 1, and let $u' = u + \epsilon\mathbb{1}_n \geq 0$. Because P has exactly one 1 in every row, it follows that $P\mathbb{1}_n = \mathbb{1}_d$, and $Pu' = Pu + \epsilon P\mathbb{1}_n = Pu + \epsilon\mathbb{1}_d \leq V_{\text{th}}$ due to the definition of ϵ, from which $u' \in \mathcal{U}$. Note that $MGu' = MGu + \epsilon MG\mathbb{1}_n$ and, because G is irreducibly diagonally dominant with positive diagonal and nonpositive off-diagonal entries, from which $G\mathbb{1}_n \geq 0$, with $G\mathbb{1}_n \neq 0$, then $\epsilon MG\mathbb{1}_n > 0$ due to $\epsilon M > 0$, and so $MGu' > MGu$. It follows that $g(u') > g(u) = g^*$ with $u' \neq u$, which contradicts (71). Therefore, u is extremal in $\mathcal{U}$, so that u^* is extremal in $\mathcal{U}$, which completes the first part of the proof.

Next, we will prove that u^* is irreducible; the proof will be by contradiction. Let $u \in \mathcal{U}$ be feasible with $g(u) = g^*$ and suppose that u is reducible, then by Lemma 9 we must have $\overline{v}(\mathcal{F}(u)) \neq u$. Let $u' = \overline{v}(\mathcal{F}(u))$, so that $\mathcal{F}(u') = \mathcal{F}(u)$ due to Lemma 8. Because $0 \leq u' \leq u$ due to Lemma 3, from which $Pu' \leq Pu$ due to $P \geq 0$, so that $u' \in \mathcal{U}$, the conditions of the lemma provide that $g(u') = g(u) = g^*$. Let $\delta = MGu - MGu'$. Note that $MGu' = MG\overline{v}(\mathcal{F}(u)) = \mathrm{emax}_{I\in\mathcal{F}(u)}(M'I) \leq MGu$, due to (13), and $MGu \neq MG\overline{v}(\mathcal{F}(u))$, due to $\overline{v}(\mathcal{F}(u)) \neq u$, from which $\delta \geq 0$ and $\delta \neq 0$. Combining this with $G^{-1}A > 0$, from (4), we have $0 < G^{-1}A\delta = u - u'$. Consequently, we have $0 \leq u' < u$, so that $Pu' < Pu \leq V_{\text{th}}$, the final step due to $P \geq 0$, P has no row with all zeros, and $u \in \mathcal{U}$, so that u' is not extremal in $\mathcal{U}$. But this contradicts the first part of the proof. It follows that u is irreducible, so that u^* is irreducible. Therefore, $\mathcal{F}(u^*)$ is maximal in $\mathcal{S}$. ■

Lemma 17: $\mathcal{F}(u_s)$ is maximal in $\mathcal{S}$.

Proof: Recall that ρ^* and u_s are well-defined and $(\rho^*, u_s) \in \mathcal{R}$, so that $\rho^* d \leq MGu_s$ and $\rho^* \geq 0$ which, because $d \geq 0$, gives $0 \leq \rho^* d \leq MGu_s$ and so u_s is feasible due to Lemma 5. We will prove that $\rho(\cdot)$ satisfies the conditions of Lemma 16, from which $\mathcal{F}(u_s)$ is maximal in $\mathcal{S}$. First, notice that for any $u, u' \in \mathcal{U}$, if $\mathcal{F}(u') = \mathcal{F}(u)$, it follows that $\rho(u') = \rho(u)$, due to (52). It remains to prove that for any $u, u' \in \mathcal{U}$, if $MGu' > MGu$, then $\rho(u') > \rho(u)$.

Let $\lambda = \min_{\forall i}(MGu'|_i - MGu|_i)/\max_{\forall i}(d_i)$ and let $\theta' = \rho(u) + \lambda$. Because $MGu' > MGu$ and $M > 0$, it follows that $\lambda > 0$ and $\theta' > \rho(u)$. Furthermore, we have $(\theta', u') \in \mathcal{R}$, because $u' \in \mathcal{U}$ and

$$\theta' d = \rho(u)d + \frac{\min_{\forall i}\big(MGu'|_i - MGu|_i\big)}{\max_{\forall i}(d_i)} d \tag{72}$$
$$\leq MGu + \min_{\forall i}\big(MGu'|_i - MGu|_i\big)\mathbb{1}_n \tag{73}$$
$$\leq MGu' \tag{74}$$

where in (73) we used $(\rho(u), u) \in \mathcal{R}$ and $d/\max_{\forall i}(d_i) \leq \mathbb{1}_n$. Therefore, we have $\rho(u') \geq \theta' > \rho(u)$, so that $\rho(\cdot)$ satisfies the conditions of Lemma 16 and $\mathcal{F}(u_s)$ is maximal in $\mathcal{S}$. ∎

Lemma 18: $\mathcal{F}(u_c)$ is maximal in $\mathcal{S}$.

Proof: Recall that ζ, ω, and u_c are well-defined and $(\zeta, \omega, u_c) \in \mathcal{C}$, so that $M'\zeta \leq MGu_c$ and $\zeta \geq 0$ which, because $M' \geq 0$, gives $0 \leq M'\zeta \leq MGu_c$ and so u_c is feasible due to Lemma 5. We will prove that $\xi(\cdot)$ satisfies the conditions of Lemma 16, from which $\mathcal{F}(u_c)$ is maximal in $\mathcal{S}$. First, notice that for any $u, u' \in \mathcal{U}$, if $\mathcal{F}(u') = \mathcal{F}(u)$, it follows that $\xi(u') = \xi(u)$, due to (58). It remains to prove that for any $u, u' \in \mathcal{U}$, if $MGu' > MGu$, then $\xi(u') > \xi(u)$.

For any $u \in \mathcal{U}$, there must exist a vector $I \in \mathcal{F}(u)$ and θ, where $0 \leq \theta d \leq MGu$, such that $\sum_{j=1}^{m} I_j + m\theta = \xi(u)$. Let $\lambda = \min_{\forall i}(MGu'|_i - MGu|_i)/\max_{\forall i,j}(m_{ij})$. Because $MGu' > MGu$ and $M > 0$, it follows that $\lambda > 0$. Also, let $e_1 \in \mathbb{R}^m$ be the vector whose first entry is 1 and all other entries are 0 and let $I' = I + \lambda e_1$. Because $\lambda > 0$, we have $\lambda e_1 \geq 0$, $\lambda e_1 \neq 0$, $I' \geq I \geq 0$, and $I' \neq I$, so that $\sum_{j=1}^{m} I'_j + m\theta > \sum_{j=1}^{m} I_j + m\theta = \xi^*$. Furthermore, we have $I' \in \mathcal{F}(u')$, because

$$M'I' = M'I + \lambda M' e_1 = M'I + \lambda c'_1 \quad (75)$$

$$= M'I + \frac{\min_{\forall i}\big(MGu'|_i - MGu|_i\big)}{\max_{\forall i,j}\big(m_{ij}\big)} c_1 \quad (76)$$

$$\leq MGu + \min_{\forall i}\big(MGu'|_i - MGu|_i\big)\mathbb{1}_n \quad (77)$$

$$\leq MGu' \quad (78)$$

where in (77) we used $I \in \mathcal{F}(u)$ and $c_1/\max_{\forall i,j}(m_{ij}) \leq \mathbb{1}_n$. Also, we have $0 \leq \theta d \leq MGu < MGu'$. Therefore, we have $I' \in \mathcal{F}(u')$, and θ satisfying $0 \leq \theta d \leq MGu'$, with $\xi(u') \geq \sum_{j=1}^{m} I'_j + m\theta > \xi(u)$, so that $\xi(\cdot)$ satisfies the conditions of Lemma 16 and $\mathcal{F}(u_c)$ is maximal in $\mathcal{S}$. ∎

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