Low-pass Filter for Computing the Transition Density in Digital Circuits

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Abstract

Estimating the power dissipation and the reliability of integrated circuits is a major concern of the semiconductor industry. Previously [1], we showed that a good measure of power dissipation and reliability is the extent of circuit switching activity, called the *transition density*. However, the algorithm for computing the density in [1] is very basic and does not take into account the effect of inertial delays of logic gates. Thus, as we will show in this paper, the transition density may be severely overestimated in high frequency applications. To overcome this problem, we model the effect of gate delay on logic signals in the form of a conceptual *low-pass filter* module that does not allow unacceptably short logic pulses to propagate. Using a stochastic model of logic signals, we then derive the equations required to propagate the transition density through the filter. We will present experimental results that illustrate the validity and importance of these results.

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1. Introduction

The dramatic decrease in feature size and the corresponding increase in the number of devices on a chip, combined with the growing demand for portable communication and computing systems, has made power consumption one of the major concerns in VLSI circuits and systems design. Indeed, excessive power dissipation in integrated circuits not only discourages their use in a portable environment, but also causes overheating, which can lead to soft errors or permanent damage. As such, power dissipation becomes one of many other *reliability* concerns (such as electromigration, hot-carrier degradation, etc) that are becoming increasingly important with today's technology.

A crucial observation is that the power dissipation and, in general, the reliability of a chip is directly related to the extent of its *switching activity*, i.e., the rate at which its nodes are switching. Less active circuits consume less power and are more reliable. However, estimating the level of activity has traditionally been very hard because it depends on the specific signals being applied to the circuit primary inputs. These signals are generally unknown during the design phase because they depend on the system in which the chip will eventually be used. Furthermore, it is practically impossible to simulate large circuits for all possible inputs. To address these issues, the *transition density* was introduced in [1] as a compact measure of switching activity in digital circuits. Simply put, the transition density at a node is the average number of transitions per second at that node, and it can be efficiently evaluated without requiring exact information about the primary input signals.

However, the algorithm for computing the density in [1] is very basic and does not take into account the effect of inertial delays of logic gates. Thus, the transition density may be severely overestimated for high speed circuits, as we will now demonstrate.

Consider a multi-input multi-output logic module M whose outputs are Boolean functions of its inputs. M may be a single gate, a cell, or a higher level module. A simplified timing model was used in [1] to represent the propagation delays through M, consisting of a single value of delay for every input-output node pair. The main result in [1] was a simple expression for the density at the outputs of M, in terms of its input densities and its Boolean difference probabilities, as follows (assuming M has n inputs x_i and m outputs y_j) :

$$D(y_j) = \sum_{i=1}^{n} P\left(\frac{\partial y_j}{\partial x_i}\right) D(x_i) \tag{1.1}$$

The resulting algorithm requires only two pieces of information about every primary input node, namely its *equilibrium probability* $P(x)$ (fraction of time that it is high) and its *transition density* $D(x)$. Thus (1.1) provides a very efficient way of *propagating* these values throughout the circuit so that $P(x)$ and $D(x)$ are computed for all internal nodes.

The problem with this approach is that it places no checks or restrictions on the *maximum* density (or, equivalently, the minimum pulse width) at a module's output nodes. This is a result of the simplified timing model. To illustrate, consider an n-input OR gate whose inputs have equal probabilities $P = 0.5$ and equal densities $D = d$. Since the Boolean difference $\frac{\partial y_j}{\partial x_i}$ is the OR of the $(n-1)$ other inputs, its probability is at least 0.5, which leads to $D(y) \geq (nd/2)$. Thus, for large enough n, the gate output will carry arbitrarily high density, and therefore unrealistically short pulses. In practice, such short pulses are *not* generated; they are *glitches* that are *filtered out* because the module is not fast enough to respond to them. In order to model this *filtration* effect of the circuit inertial delays, we introduce a new delay block called a *filter block* at every module output, as shown in Fig. 1.

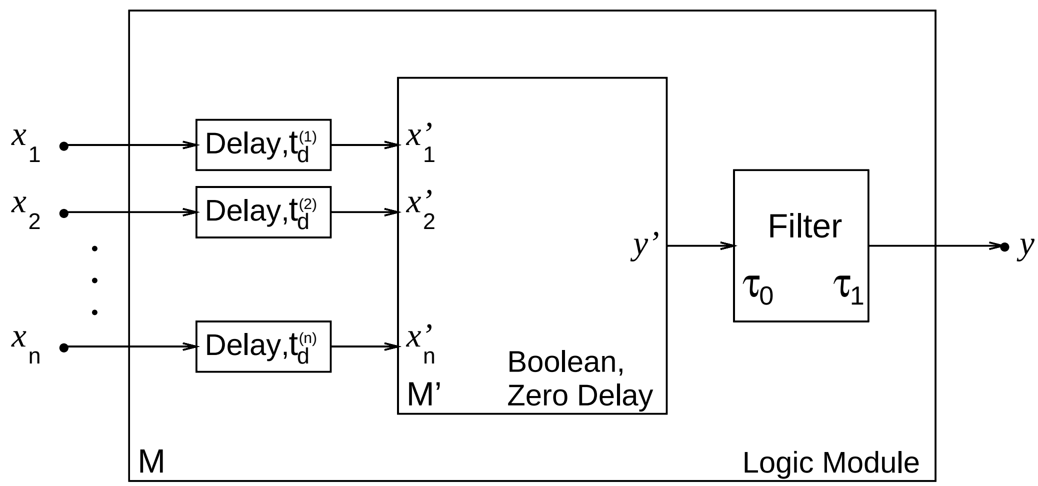

Figure 1. Block diagram of timing model.

The internal block M' is a zero-delay Boolean block that has the same Boolean function as M. The delay blocks, which can have different t_d delay values, simply introduce a delay in the logic signal which is the same for a rising or falling transition. In the general case of a multi-output module, these delays may be different for different output nodes.

The *filter* block is a delay block with a *low-pass filtering* property, that may be defined as follows : a $0 \rightarrow 1$ $(1 \rightarrow 0)$ transition at the filter input is transmitted to its output after a delay of τ_1 (τ_0) if and only if its input does not change state during that time. Thus the filter block effectively sets a *minimum pulse width* at the output y : the minimum high (low) pulse width at the module output is τ_0 (τ_1).

This paper is devoted to the analysis of a filter block, in order to propagate $P(x)$ and $D(x)$ from its input to its output. In section 2, we review the formalism of a *companion process* of a logic signal that was introduced in [1]. The following section discusses the distribution of pulse widths of a logic signal and makes a simplifying assumption that is required for the remainder of this paper. In section 4, we study the behavior of a filter block and present our main result on the propagation of density values through it. The following two sections are devoted to experimental results and conclusions. Finally, some proofs relevant to the simplifying assumption are given in appendix A, and the proof of our main result is given in appendix B.

2. Companion Process of a Logic signal

Throughout this paper, we will use **bold font** to represent random quantities, and will denote the probability of an event A by $\mathcal{P}\{A\}$. Furthermore, if $\mathbf{x}$ is a random variable, we will denote its mean (or expected value) by $E[\mathbf{x}]$. In this section, we will briefly review the concept of a *companion process* from [1].

Let $x(t)$, $t \in (-\infty, +\infty)$, be a logic signal, i.e., it is a function of time that takes the values 0 or 1. A *0-1 stochastic process* is a stochastic process [2] that takes only 0 & 1 values. We associate with $x(t)$ a 0-1 stochastic process $\mathbf{x}(t)$, called its companion process, defined as $\mathbf{x}(t) \triangleq x(t+\boldsymbol{\tau})$ where $\boldsymbol{\tau}$ is uniform over the whole real line [1]. It was shown in [1] that $\mathbf{x}(t)$ is strict-sense stationary (for brevity : *stationary*) and mean-ergodic.

Let $n_x(T)$ be the number of transitions of $x(t)$ in the interval $(\frac{-T}{2}, \frac{+T}{2}]$. The *equilibrium probability* $P(x)$ and *transition density* $D(x)$ were defined in [1] as follows :

$$P(x) \triangleq \lim_{T\to\infty} \frac{1}{T} \int_{\frac{-T}{2}}^{\frac{+T}{2}} x(t)dt \qquad \text{and} \qquad D(x) \triangleq \lim_{T\to\infty} \frac{n_x(T)}{T} \tag{2.1}$$

The signal $x(t)$ is composed of an alternating sequence of high (corresponding to $x = 1$) and low ($x = 0$) pulses. With μ_1 (μ_0) defined as the average high (low) pulse-width of $x(t)$, it was shown in [1] that (with $\mu_0 + \mu_1 > 0$) :

$$P(x) = \frac{\mu_1}{\mu_0 + \mu_1} \qquad \text{and} \qquad D(x) = \frac{2}{\mu_0 + \mu_1} \tag{2.2}$$

Finally, with $\mathbf{n}_{\mathbf{x}}(T)$ defined as the number of transitions of $\mathbf{x}(t)$ in the interval $(\frac{-T}{2}, \frac{+T}{2}]$, it was also shown in [1] that, for any t and any $T > 0$, we have :

$$P(x) = \mathcal{P}\{\mathbf{x}(t) = 1\} \qquad \text{and} \qquad D(x) = E\left[\frac{\mathbf{n}_{\mathbf{x}}(T)}{T}\right] \tag{2.3}$$

Based on these results, the expression (1.1) was derived [1] and used to propagate probability and density values throughout the circuit.

3. Pulse Width Distributions

The purpose of this section is to introduce a simplifying assumption that will make possible the solution of the filter block in section 4. In order to illustrate the need for this assumption, and to show that it is actually *mild* in the sense that it is approximately true in practice, we will lead up to it by discussing the distribution of pulse widths in a logic signal.

In general, a logic signal $x(t)$ can have an infinite number of high pulses (corresponding to $x = 1$) and low pulses ($x = 0$). The width of a high pulse of $x(t)$ can be any value in the interval $(0, +\infty]$. Let the population of high pulse-widths have the *cumulative distribution function* (cdf) $F_1(t)$. Thus $F_1(t)$ is the fraction of high pulses that are shorter than or equal to t. Let $\mathbf{r_1}$ be a random variable with the same distribution $F_1(t)$, i.e., $\mathcal{P}\{\mathbf{r_1} \le t\} = F_1(t)$, and the *probability density function* (pdf) $f_1(t) = \frac{d}{dt}F_1(t)$. Likewise, let $\mathbf{r_0}$ be a random variable distributed as the low pulses of $x(t)$, with the cdf $F_0(t) = \mathcal{P}\{\mathbf{r_0} \le t\}$, and the pdf $f_0(t) = \frac{d}{dt}F_0(t)$.

Consider the situation shown in Fig. 2, where t_0 is a fixed (non-random) time point, and where we focus on the case $\mathbf{x}(t_0) = 0$. In this case, let the length of the low pulse around t_0 be $\hat{\mathbf{r}}_\mathbf{0}$. It is interesting to note that $\hat{\mathbf{r}}_\mathbf{0}$ does *not* have the same distribution as $\mathbf{r_0}$ (this is an instance of the so-called *inspection paradox*, see [5], page 69). This happens because it is more probable for $(t_0 + \boldsymbol{\tau})$ to lie in the longer pulses of $x(t)$ (recall that $\mathbf{x}(t_0) = x(t_0 + \boldsymbol{\tau})$, and $\boldsymbol{\tau}$ is uniform over the whole real line).

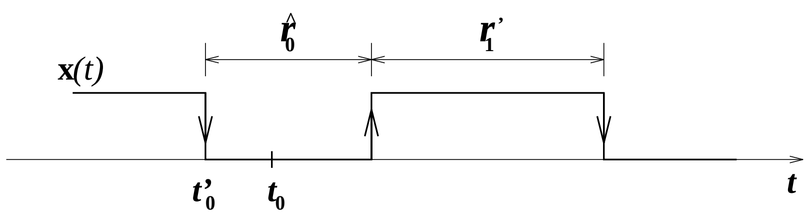

Figure 2. Low pulse width around fixed $t = t_0$.

Let $\hat{F}_0(t)$ and $\hat{f}_0(t)$ be the cdf and pdf of $\hat{\mathbf{r}}_\mathbf{0}$, respectively. We show in appendix A that these distributions are given by :

$$\hat{f}_0(t) = \frac{t}{\mu_0} f_0(t) \qquad \text{and} \qquad \hat{F}_0(t) = \int_0^t \frac{z}{\mu_0} f_0(z) dz \tag{3.6a, b}$$

Likewise, when $\mathbf{x}(t_0) = 1$, if the length of the high pulse around t_0 is $\hat{\mathbf{r}}_\mathbf{1}$, with a cdf of $\hat{F}_1(t)$ and a pdf of $\hat{f}_1(t)$, it can be shown that :

$$\hat{f}_1(t) = \frac{t}{\mu_1} f_1(t) \qquad \text{and} \qquad \hat{F}_1(t) = \int_0^t \frac{z}{\mu_1} f_1(z) dz \tag{3.7a, b}$$

Going back to the $\mathbf{x}(t_0) = 0$ situation shown in Fig. 2, let $\mathbf{r}'_\mathbf{1}$ be the length of the first high pulse to the right of $\hat{\mathbf{r}}_\mathbf{0}$, as shown in the figure. We show in appendix A that, if $\mathbf{r}'_\mathbf{1}$ and $\hat{\mathbf{r}}_\mathbf{0}$ are *independent*, then $\mathbf{r}'_\mathbf{1}$ is distributed as $\mathbf{r_1}$.

We can generalize this result, so that if $\mathbf{r}'_\mathbf{1}$ is *any* high pulse to the right or left of $\hat{\mathbf{r}}_\mathbf{0}$, and if it is *independent* of $\hat{\mathbf{r}}_\mathbf{0}$, then $\mathbf{r}'_\mathbf{1}$ is distributed as $\mathbf{r_1}$. Likewise, when $\mathbf{x}(t_0) = 1$, if $\mathbf{r}'_\mathbf{0}$ is *any* high pulse to the right or left of $\hat{\mathbf{r}}_1$, and if it is *independent* of $\hat{\mathbf{r}}_1$, then $\mathbf{r}_0'$ is distributed as $\mathbf{r}_0$. Otherwise, i.e., if these pulses are not independent, then their distributions will depend on their correlation with $\hat{\mathbf{r}}_0$ or $\hat{\mathbf{r}}_1$. (For the interested reader : the reason that these pulses are not all distributed according to $F_1(t)$ or $F_0(t)$ is because the fixed time reference t_0 was chosen arbitrarily, and not, for instance, at the beginning of a pulse. This choice will become important when Fig. 2 is again invoked in the derivations in the appendices.)

In practice, it is reasonable to assume for a general logic signal that two pulses that are sufficiently separated in time will be uncorrelated or independent. By extending this intuitive notion, we arrive at the following simplifying assumption which will make possible the solution of a filter block in the next section :

Assumption: *The width of every pulse of* $\mathbf{x}(t)$ *is independent of all other pulse widths in its past* **and** *future.*

This assumption is *mild* in the sense that it is approximately true in practice, such as when two pulses are widely separated in time.

Since the collection of all previous (future) pulse widths completely determines the past (future) of $\mathbf{x}(t)$, then every pulse of $\mathbf{x}(t)$ is independent of its past *and* future. As a result, the process $\mathbf{x}(t)$ (probabilistically) restarts itself after every transition. A stochastic process with this property is commonly called a *stationary, time-reversible, semi-Markov 0-1 process* [5]. We will make use of these properties in the next section as we study the propagation through a filter block.

4. Filter Block Analysis

Let F be a filter block with input $x(t)$ and output $y(t)$. The behavior of a filter can be formally defined by the state diagram shown in Fig. 3. A filter has four states, determined by the current values of x and y. The states S_0 (corresponding to $x = y = 0$) and S_3 (corresponding to $x = y = 1$) are called *stable states.* The filter will stay in these states indefinitely if x does not change. The states S_1 $(x = 0, y = 1)$ and S_2 $(x = 1, y = 0)$ are called *unstable states.* If the filter gets into state S_1 (S_2), then it can stay there for at most τ_0 (τ_1), after which time it will automatically transition to the stable state S_0 (S_3). Transitions of $y(t)$ are generated only during these autonomous transitions from an unstable to a stable state. If the filter is in an unstable state, and a transition at x occurs, then it will move to a stable state immediately, and no transition at y will be generated.

The main result of this paper is the following theorem that shows how $P(y)$ and $D(y)$ can be computed from $P(x)$ and $D(x)$:

Theorem 1. *For a filter with input x and output y, and given the basic assumption made*

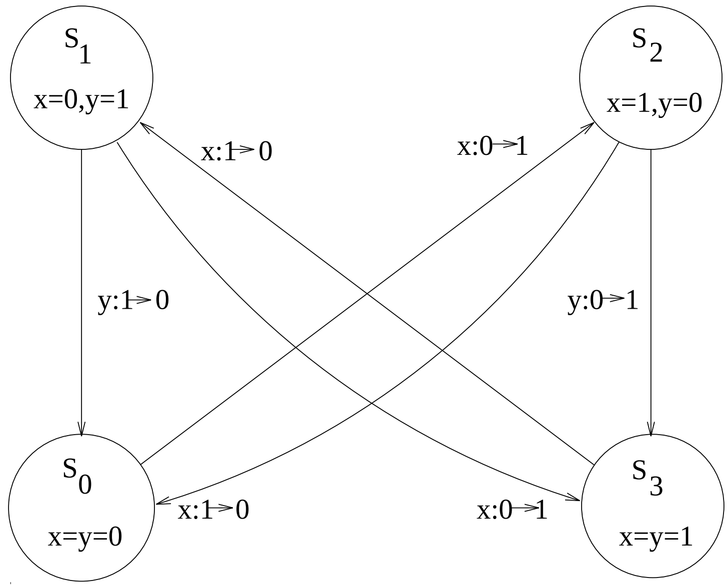

Figure 3. State diagram of a filter block.

above, we have :

$$P(y) = P(x) - \left\{ \hat{F}_1(\tau_1) + \frac{\tau_1}{\mu_1}\left[1 - F_1(\tau_1)\right] \right\} \frac{\left[1 - F_0(\tau_0)\right]}{\left[1 - F_0(\tau_0)F_1(\tau_1)\right]} P(x)$$
$$+ \left\{ \hat{F}_0(\tau_0) + \frac{\tau_0}{\mu_0}\left[1 - F_0(\tau_0)\right] \right\} \frac{\left[1 - F_1(\tau_1)\right]}{\left[1 - F_0(\tau_0)F_1(\tau_1)\right]} \left[1 - P(x)\right] \tag{4.1}$$

and :

$$D(y) = \frac{\left[1 - F_0(\tau_0)\right]\left[1 - F_1(\tau_1)\right]}{1 - F_0(\tau_0)F_1(\tau_1)} D(x) \tag{4.2}$$

Proof : See appendix B. ■

Notice that all that is needed to use these results are the two distribution functions $F_0(t)$ and $F_1(t)$. In practice, it is not clear what these functions should be, or how one might estimate them. We will have more to say on this in the next section. For now, we will show that by using a simple approximation, we can simplify this requirement so that only $F_0(\tau_0)$ and $F_1(\tau_1)$ are required, as follows. The ratio $D(y)/D(x)$ will be called the *transmission probability* of the filter F, and will be denoted by P_F :

$$P_F = \frac{\left[1 - F_0(\tau_0)\right]\left[1 - F_1(\tau_1)\right]}{1 - F_0(\tau_0)F_1(\tau_1)} \tag{4.3}$$

Using this notation, we can (after some algebraic manipulation) rewrite (4.1) and (4.2) as :

$$P(y) = P_F \times \left\{ \frac{1}{1 - F_0(\tau_0)} - P(x) + \left(E[r_1 - \tau_1 \mid r_1 > \tau_1] - E[r_0 - \tau_0 \mid r_0 > \tau_0]\right) \frac{D(x)}{2} \right\} \tag{4.4}$$

and :

$$D(y) = P_F \times D(x) \tag{4.5}$$

Furthermore, for small τ_1 and τ_0, one can use the approximations :

$$E[r_1 - \tau_1 \mid r_1 > \tau_1] \approx \frac{(\mu_1 - \tau_1) + \tau_1 F_1(\tau_1)/2}{1 - F_1(\tau_1)} \tag{4.6}$$

$$E[r_0 - \tau_0 \mid r_0 > \tau_0] \approx \frac{(\mu_0 - \tau_0) + \tau_0 F_0(\tau_0)/2}{1 - F_0(\tau_0)} \tag{4.7}$$

which are obtained by setting $E[\tau_1 - r_1 \mid r_1 \leq \tau_1] \approx \tau_1/2$, for small τ_1, and likewise for τ_0. Thus all that is needed is $F_0(\tau_0)$ and $F_1(\tau_1)$. The experimental results in the next section will not use these approximations, but will be based on the accurate expressions (4.1) and (4.2).

5. Experimental Results and Discussion

Given the equilibrium probability $P(x)$ and transition density $D(x)$ at the primary inputs of a combinational logic circuit, one can compute the corresponding probability and density at every internal node using (1.1). The density values can be used to estimate the circuit power dissipation as well as the susceptibility to certain reliability problems such as hot-carrier degradation and electromigration. The density propagation algorithm based on (1.1) was implemented in the program *densim* and presented in [1]. The results of this paper (equations (4.1) and (4.2)) have been incorporated into densim by simply applying the filter operation to the output of every gate as shown in the block diagram in Fig. 1. The filter parameters τ_0 and τ_1 can be set by the user in the module library; otherwise, they are derived from the propagation delay and rise/fall times of a module.

In order to use the filter equations (4.1) and (4.2), however, we need to know the pulse width distribution functions $F_1(t)$ and $F_2(t)$. The *form* of these distributions is generally unknown, but one may make reasonable assumptions about them, as follows. We will again make use of the intuitive property that the values of a logic signal at widely separated time points are relatively independent. If we extend this property to the point that the future value of the signal is independent of its past, once its present value is specified, then it is said to be Markov [2] and its high and low pulses are known to be *exponentially distributed.* In the absence of any other information, therefore, it seems that the exponential distribution is a reasonable assumption. We will come back to this point later in this section, after we've considered the effect of these distributions on the filter behavior.

If the mean pulse width is μ, then the probability density function (pdf) for an exponential distribution is $(1/\mu)e^{-t/\mu}$ and is shown in Fig. 4a. This distribution is a special case of the *gamma* distribution - it is a gamma distribution of order 1. Two other gamma distributions, of orders 2 and 3, are also shown in the same figure.

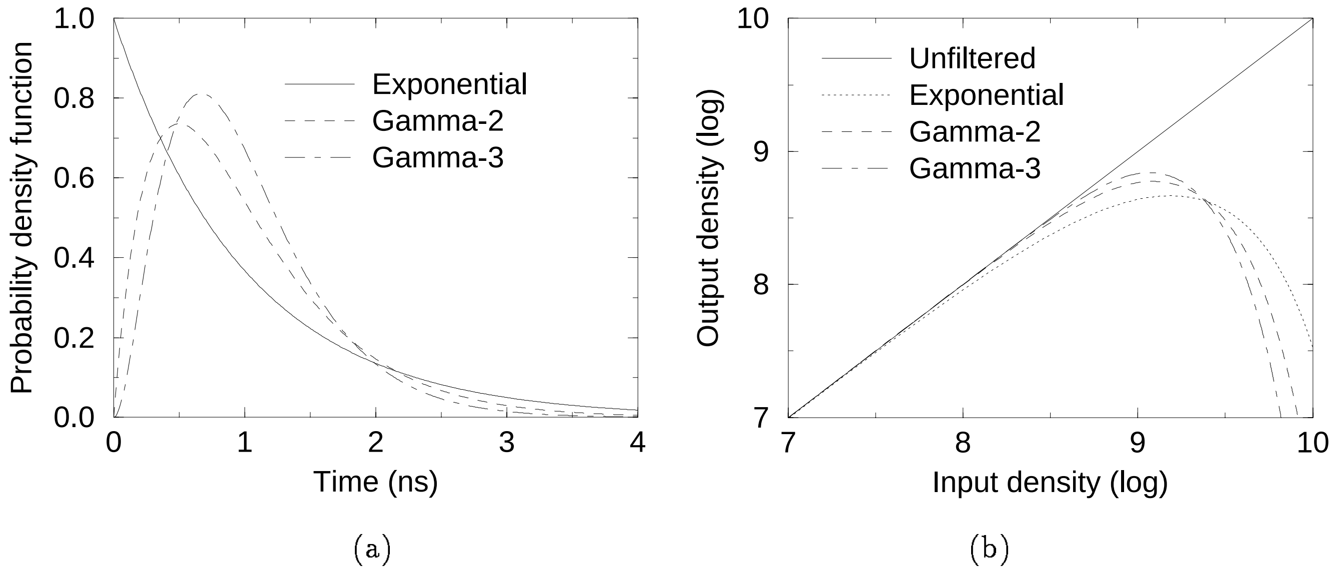

Figure 4. Three distributions (a) with $\mu = 1$ nsec and the corresponding filter transfer characteristics (b) with $\tau_0 = \tau_1 = 0.5$ nsec and $P(x) = 0.5$.

The effect of the filter on an input waveform for the three distributions is shown in Fig. 4b. The density of the filtered signal starts to deviate appreciably from that of the unfiltered signal at high densities. This plot was obtained using equations (4.1) and (4.2).

It is prudent at this point to experimentally validate the results of theorem 1 and of Fig. 4b. To do this, we applied a randomly generated logic signal to the inputs of a filter block, and processed the signal as one would in a logic simulator. We then monitored the signal at the filter output. Averaging over a long enough simulation time, the output probability and density should converge to those predicted by theorem 1. This behavior was indeed observed, for the three different distributions, as shown in Figs. 5 and 6. In both figures (and in the remainder of this section), the results of logic simulation are marked "logsim," while the results of applying theorem 1 are marked "densim."

Going back to the issue of the form of the distributions $F_0(t)$ and $F_1(t)$, we have performed extensive experimental studies on several kinds of circuits, but there seems to be no general statements that one can make about the shape of the distributions in practice. Fortunately, though, we have found that the overall power dissipation of a circuit (a weighted average of the node densities) is *relatively insensitive* to the pulse width distributions at its primary inputs. For instance the average power dissipated in a 32-bit ripple adder (measured with a logic simulation, with an input density of 2×10^9) was found to be 17.74 mW for the exponential distribution, 17.49 mW for a gamma distribution of order 2, and 17.33 mW for a gamma distribution of order 3. Therefore, at least for purposes of computing the average power, it is enough to assume some arbitrary input pulse width distribution. For the reasons

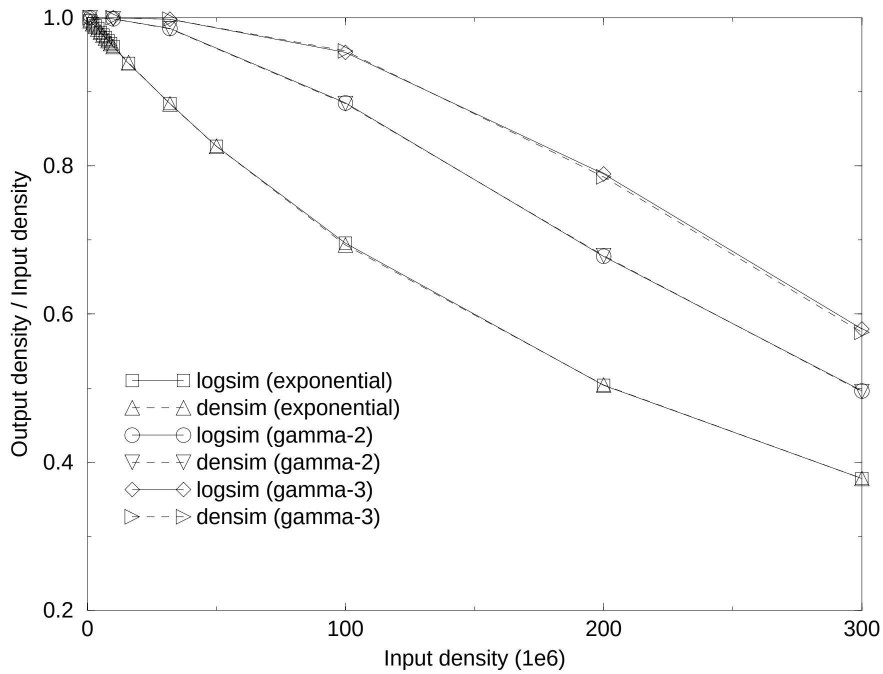

Figure 5. Filter transfer characteristics with $\tau_0 = \tau_1 = 1$ nsec and $P(x) = 0.5$ for different input distributions.

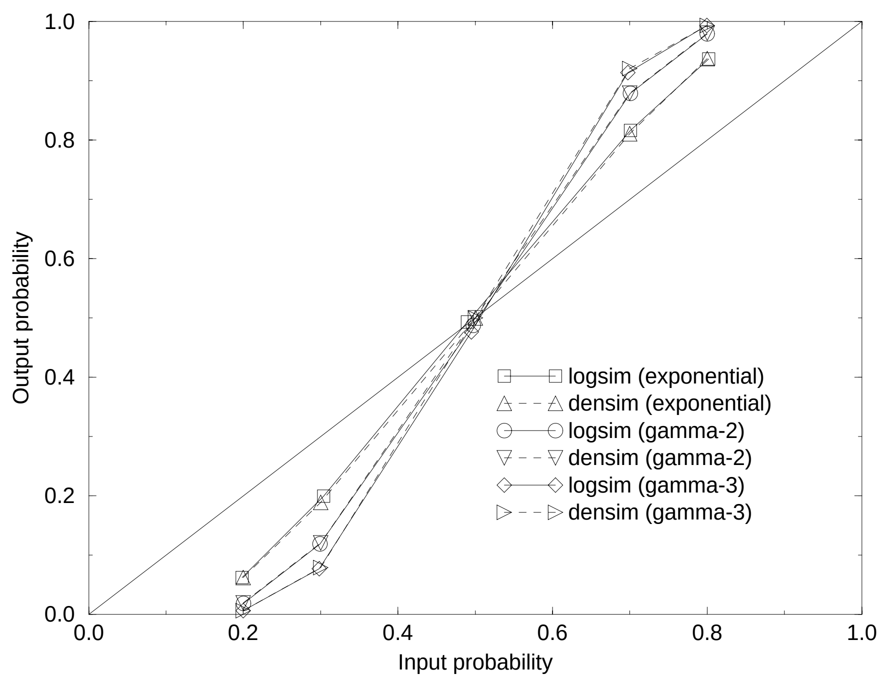

Figure 6. Filter transfer characteristics with $\tau_0 = \tau_1 = 1$ nsec and $D(x) = 1.2e9$ (600 MHz) for different input distributions.

given above, we have chosen to use the exponential distribution. We should point out that the 2×10^9 input density chosen for this test case is high enough for the filter mechanism to "make a difference" in the results, as can be seen from Fig. 7.

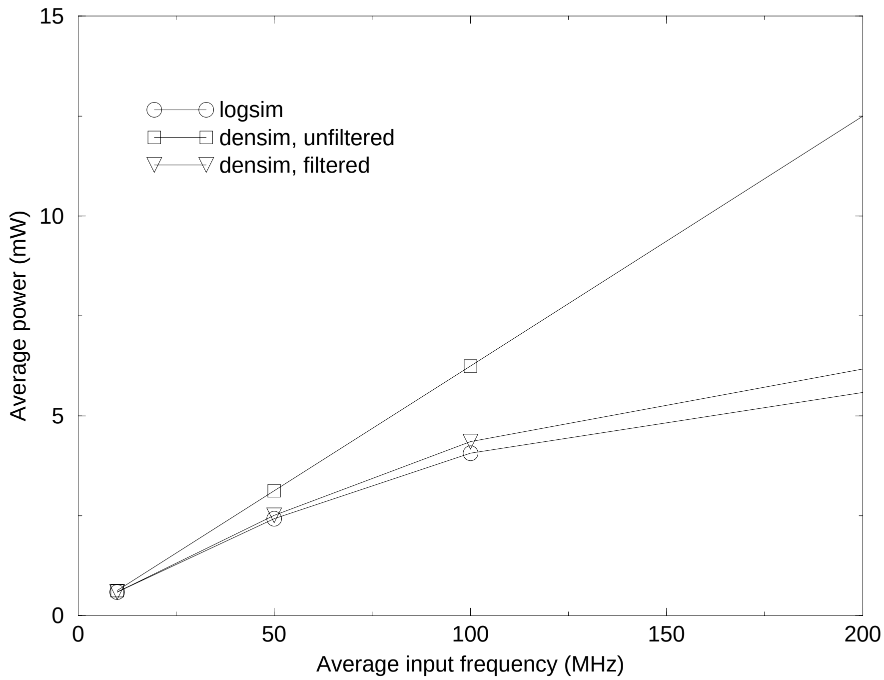

Figure 7. Results for a 32-bit ripple adder with $P(x) = 0.5$ for all inputs.

The plot shown in Fig. 7 compares the average power dissipation of the circuit, as measured by logic simulation, to that measured by densim with and without the filter mechanism. The horizontal axis shows the *average frequency* of the signals applied to the circuit primary inputs (the transition density is twice the average frequency [1]). It clearly shows the need for the filter mechanism at higher frequencies. Fig. 8 shows the results of a similar analysis for a 4-bit parallel multiplier and a 4-bit alu. Finally, some more results are shown in Fig. 9 for the first two ISCAS-85 benchmark circuits.

The above experimental results demonstrate the validity of the results in theorem 1, and the fact that if the filter mechanism is not used, then the basic density propagation algorithm (1.1) will severely deviate from the correct results at higher frequencies.

As a final note, we should say that the improved accuracy afforded by the filter mechanism is obtained at virtually no speed penalty. Equations (4.1) and (4.2) have to be evaluated only once for a logic gate. Thus, the density propagation algorithm remains as efficient as was shown in [1].

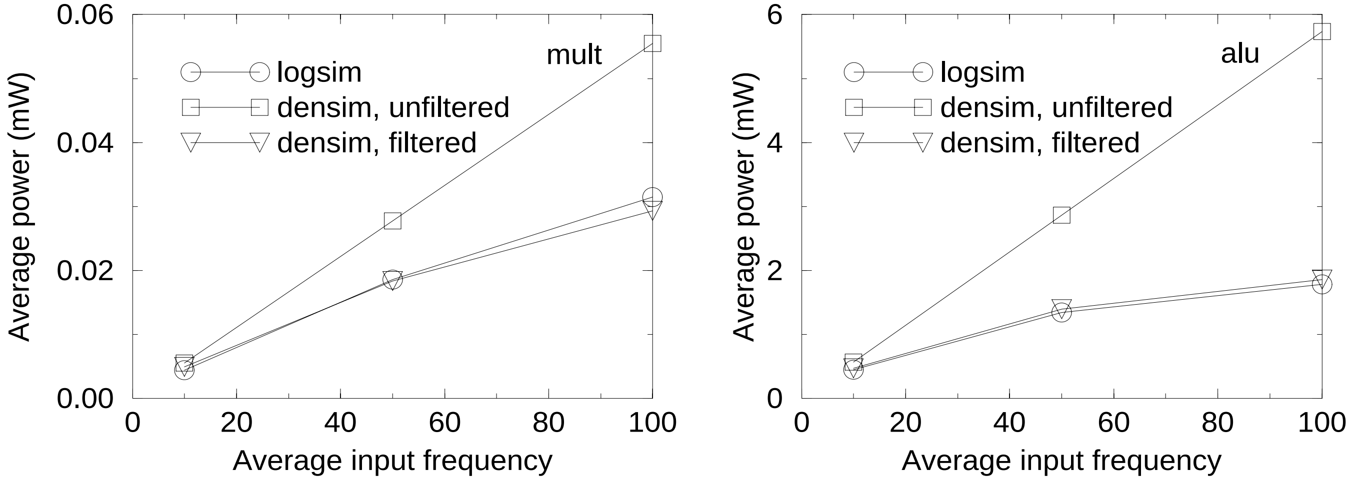

Figure 8. Results for a 4-bit parallel multiplier and a 4-bit alu with $P(x) = 0.5$ for all inputs.

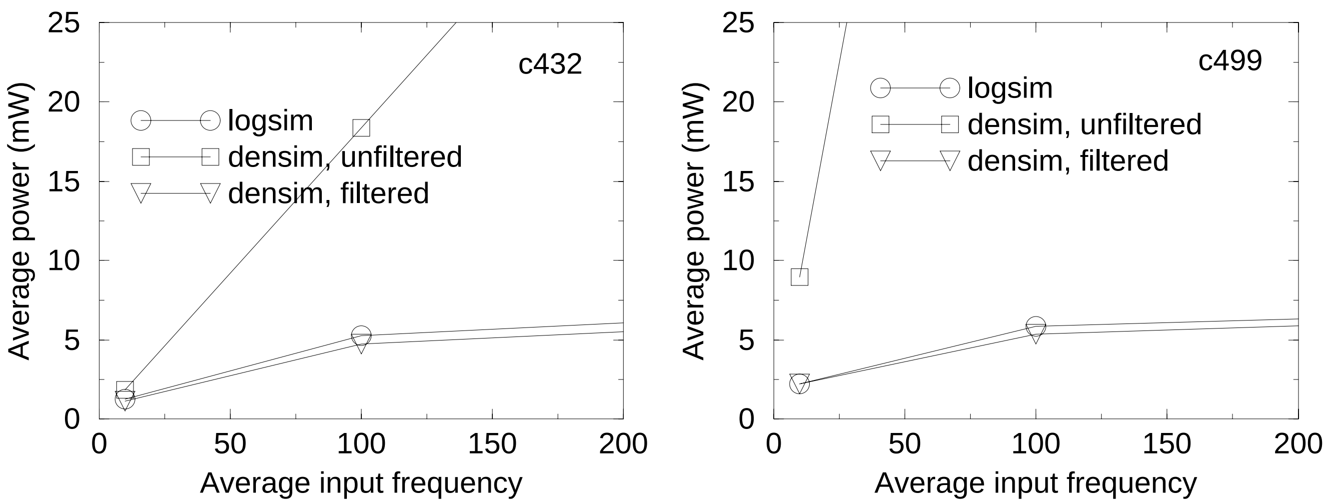

Figure 9. Results for c432 and c499 with $P(x) = 0.5$ for all inputs.

On the other hand, the overall approach still has some accuracy problems, even at low frequencies, due to the independence assumptions implicit in (1.1). As was discussed in [1], this is due to node correlations resulting from reconvergent fanout. This issue is part of our continuing work in this area.

6. Summary and Conclusions

The average number of logic transitions per second, called the *transition density*, was introduced in [1] as a measure of circuit power dissipation and reliability. An algorithm was also presented to compute the node densities by propagating the transition densities specified at the circuit primary inputs. In this paper, we have pointed out that that algorithm does not place any checks or restrictions on the maximum transition density at a node. Realistically, an upper bound on the node density does exist because a logic gate with non-zero delay cannot propagate arbitrarily short logic pulses. Pulses that are too short appear as glitches and do not propagate through the gate.

In order to overcome this problem, we have presented an extension to the transition density approach in [1] by taking into account the effect of the inertial delay of a logic gate. In the framework of the stochastic representation of logic signals of [1], we have modeled this effect with a conceptual low-pass filter block. Detailed analysis of this block has yielded compact expressions for the transition density at its output given the density at its input.

Experimental results demonstrate that the filter module behaves as it should, and that the filter mechanism is required in order to maintain accuracy at higher frequencies.

Appendix A
Some Proofs Relevant to Section 2

Recall that $\mathbf{r_1}$ is a random variable distributed as the high pulses of $x(t)$, with the *cumulative distribution function* (cdf) $F_1(t) = \mathcal{P}\{\mathbf{r_1} \leq t\}$, and the *probability density function* (pdf) $f_1(t) = \frac{d}{dt}F_1(t)$. Likewise, $\mathbf{r_0}$ is a random variable distributed as the low pulses of $x(t)$, with the cdf $F_0(t) = \mathcal{P}\{\mathbf{r_0} \leq t\}$, and the pdf $f_0(t) = \frac{d}{dt}F_0(t)$.

In an interval $(\frac{-T}{2}, \frac{+T}{2}]$, let $n_{x,0}(T)$ be the total number of low pulses of $x(t)$ and $n^t_{x,0}(T)$ be the number of those low pulses whose width is in the interval $[t, t+dt]$. From the definition of a pdf, it follows that :

$$f_0(t)dt = \mathcal{P}\{t \leq \mathbf{r_0} \leq t + dt\} = \lim_{T\to\infty} \frac{n^t_{x,0}(T)}{n_{x,0}(T)} \qquad (A.1)$$

Recall that, in the definition of the companion process $\mathbf{x}(t)$, $\boldsymbol{\tau}$ is a random variable uniformly distributed over the whole real line (time axis). Thus, for any fixed t_0, $\mathbf{x}(t_0) = x(t_0 + \boldsymbol{\tau})$ is a random variable equal to either 0 or 1. If $\mathbf{x}(t_0) = 0$, let the length of the low pulse around t_0 be $\hat{\mathbf{r}}_\mathbf{0}$, as shown in Fig. 2. It is interesting to note that, as we will now show, $\hat{\mathbf{r}}_\mathbf{0}$ does *not* have the same distribution as $\mathbf{r_0}$ (this is an instance of the so-called *inspection paradox*, see [5], page 69). This happens because it is more probable for $(t_0 + \boldsymbol{\tau})$ to lie in the longer pulses of $x(t)$.

Let $\mathcal{R}_0 \triangleq \{t : x(t) = 0\}$ be the subset of the time axis for which $x = 0$, and $\mathcal{R}^t_0 \subseteq \mathcal{R}_0$ be the set of those $x = 0$ intervals whose width is in $[t, t + dt]$. Let $\hat{F}_0(t)$ and $\hat{f}_0(t)$ be the cdf and pdf of $\hat{\mathbf{r}}_\mathbf{0}$, respectively. From the definition of a pdf, it follows that :

$$\hat{f}_0(t)dt = \mathcal{P}\{t \leq \hat{\mathbf{r}}_\mathbf{0} \leq t + dt\} = \mathcal{P}\{t_0 + \boldsymbol{\tau} \in \mathcal{R}^t_0 \mid t_0 + \boldsymbol{\tau} \in \mathcal{R}_0\} = \frac{\mathcal{P}\{\boldsymbol{\tau} \in \mathcal{R}^t_0\}}{\mathcal{P}\{\boldsymbol{\tau} \in \mathcal{R}_0\}} \qquad (A.2)$$

where the last equality follows because $t_0+\boldsymbol{\tau}$ has the same distribution as $\boldsymbol{\tau}$ [1] and $\mathcal{R}_0^t \subseteq \mathcal{R}_0$. If $\boldsymbol{\tau}_T$ is uniform over $(\frac{-T}{2}, \frac{+T}{2}]$ with the cdf $F_{\boldsymbol{\tau}_T}(t) = 1/T$, for $t \in (\frac{-T}{2}, \frac{+T}{2}]$, then :

$$\mathcal{P}\{\boldsymbol{\tau} \in \mathcal{R}_0\} = \lim_{T\to\infty} \mathcal{P}\{\boldsymbol{\tau}_T \in \mathcal{R}_0\} = \lim_{T\to\infty} \frac{\mu_0 n_{x,0}(T)}{T} \tag{A.3}$$

and :

$$\mathcal{P}\{\boldsymbol{\tau} \in \mathcal{R}_0^t\} = \lim_{T\to\infty} \mathcal{P}\{\boldsymbol{\tau}_T \in \mathcal{R}_0^t\} = \lim_{T\to\infty} \frac{t n_{x,0}^t(T)}{T} \tag{A.4}$$

Therefore :

$$\hat{f}_0(t)dt = \lim_{T\to\infty} \frac{t n_{x,0}^t(T)}{\mu_0 n_{x,0}(T)} = \frac{t}{\mu_0} f_0(t)dt \tag{A.5}$$

which leads to :

$$\hat{f}_0(t) = \frac{t}{\mu_0} f_0(t) \qquad \text{and} \qquad \hat{F}_0(t) = \int_0^t \frac{z}{\mu_0} f_0(z)dz \tag{A.6a, b}$$

Likewise, when $\mathbf{x}(t_0) = 1$, if the length of the high pulse around t_0 is $\hat{\mathbf{r}}_1$, with a cdf of $\hat{F}_1(t)$ and a pdf of $\hat{f}_1(t)$, it can be shown that :

$$\hat{f}_1(t) = \frac{t}{\mu_1} f_1(t) \qquad \text{and} \qquad \hat{F}_1(t) = \int_0^t \frac{z}{\mu_1} f_1(z)dz \tag{A.7a, b}$$

Going back to the $\mathbf{x}(t_0) = 0$ situation shown in Fig. 2, let $\mathbf{r}_1'$ be the length of the first high pulse to the right of $\hat{\mathbf{r}}_0$, as shown in the figure. We will now show that, if $\mathbf{r}_1'$ and $\hat{\mathbf{r}}_0$ are *independent*, then $\mathbf{r}_1'$ is distributed as $\mathbf{r}_1$.

Let $\mathcal{R}_0^{t'}$ be the set of all $x = 0$ time intervals whose neighboring (to the right) high pulses have widths in $[t, t+dt]$. If $f_1'(t)$ is the cdf of $\mathbf{r}_1'$, then using the same development made above :

$$f_1'(t)dt = \mathcal{P}\{t \le \mathbf{r}_1' \le t+dt\} = \mathcal{P}\{t_0+\boldsymbol{\tau} \in \mathcal{R}_0^{t'} \mid t_0+\boldsymbol{\tau} \in \mathcal{R}_0\} = \frac{\mathcal{P}\{\boldsymbol{\tau} \in \mathcal{R}_0^{t'}\}}{\mathcal{P}\{\boldsymbol{\tau} \in \mathcal{R}_0\}} \tag{A.8}$$

$$\mathcal{P}\{\boldsymbol{\tau} \in \mathcal{R}_0\} = \lim_{T\to\infty} \mathcal{P}\{\boldsymbol{\tau}_T \in \mathcal{R}_0\} = \lim_{T\to\infty} \frac{\mu_0 n_{x,0}(T)}{T} \tag{A.9}$$

$$\mathcal{P}\{\boldsymbol{\tau} \in \mathcal{R}_0^{t'}\} = \lim_{T\to\infty} \mathcal{P}\{\boldsymbol{\tau}_T \in \mathcal{R}_0^{t'}\} = \lim_{T\to\infty} \frac{\mu_0^{t'} n_{x,0}^t(T)}{T} \tag{A.10}$$

where $\mu_0^{t'}$ is the mean of low pulses whose neighboring (to the right) high pulses have widths in $[t, t+dt]$. However, since $\hat{\mathbf{r}}_0$ and $\mathbf{r}_1'$ are independent, we must have $\mu_0^{t'} = \mu_0$. Therefore :

$$f_1'(t)dt = \lim_{T\to\infty} \frac{\mu_0^{t'} n_{x,1}^t(T)}{\mu_0 n_{x,0}(T)} = \lim_{T\to\infty} \frac{n_{x,0}^t(T)}{n_{x,0}(T)} = f_1(t)dt \tag{A.11}$$

Thus $\mathbf{r}'_1$ is distributed as $\mathbf{r}_1$. We can generalize this result, so that if $\mathbf{r}'_1$ is *any* high pulse to the right or left of $\hat{\mathbf{r}}_0$, and if it is *independent* of $\hat{\mathbf{r}}_0$, then $\mathbf{r}'_1$ is distributed as $\mathbf{r}_1$. Likewise, when $\mathbf{x}(t_0) = 1$, if $\mathbf{r}'_0$ is *any* high pulse to the right or left of $\hat{\mathbf{r}}_1$, and if it is *independent* of $\hat{\mathbf{r}}_1$, then $\mathbf{r}'_0$ is distributed as $\mathbf{r}_0$.

Otherwise, i.e., if these pulses are not independent, then their distributions will depend on their correlation with $\hat{\mathbf{r}}_0$ and $\hat{\mathbf{r}}_1$. Since the collection of all previous (future) pulse widths completely determines the past (future) of $\mathbf{x}(t)$, then every pulse of $\mathbf{x}(t)$ is independent of its past *and* future. As a result, the process $\mathbf{x}(t)$ probabilistically restarts itself after every transition. A stochastic process with this property is commonly called a stationary, time-reversible, semi-Markov 0-1 process [5].

Appendix B
Proof of Theorem 1

Let $\mathbf{x}(t_0) = \uparrow$ denote the event $\{\mathbf{x}(t)$ makes a $0 \rightarrow 1$ transition at time $t_0\}$, and $\mathbf{x}(t_0) = \downarrow$ denote the event $\{\mathbf{x}(t)$ makes a $1 \rightarrow 0$ transition at time $t_0\}$. We start out with a lemma that gives the probability that the output $\mathbf{y}$ of the filter is 0 at a time when its input $\mathbf{x}$ makes a $0 \rightarrow 1$ transition.

Lemma 1. *If t_0 is any fixed time, then :*

$$\mathcal{P}\{\mathbf{y}(t_0) = 0 \mid \mathbf{x}(t_0) = \uparrow\} = \frac{1 - F_0(\tau_0)}{1 - F_0(\tau_0)F_1(\tau_1)} \qquad (B.1)$$

Proof : Since companion processes are stationary [1], then $\mathcal{P}\{\mathbf{y}(t_0) = 0 \mid \mathbf{x}(t_0) = \uparrow\}$ does not depend on t_0. Therefore, for any fixed t :

$$\mathcal{P}\{\mathbf{y}(t_0) = 0 \mid \mathbf{x}(t_0) = \uparrow\} = \mathcal{P}\{\mathbf{y}(t) = 0 \mid \mathbf{x}(t) = \uparrow\}, \quad \forall t \qquad (B.2)$$

Let $\mathbf{t}_{-1} \leq t_0$ be the (random) time of the last $0 \rightarrow 1$ transition of $\mathbf{x}(t)$ before t_0, and $\mathbf{t}'_{-1}$ be the (random) time of the $1 \rightarrow 0$ transition of $\mathbf{x}(t)$ that lies between $\mathbf{t}_{-1}$ and t_0.

If $\mathbf{y}$ is 0 at the end of a 0-pulse, $[\mathbf{t}'_{-1}, t_0]$, of $\mathbf{x}(t)$, then either that pulse persisted long enough (i.e., $t_0 - \mathbf{t}'_{-1} > \tau_0$), or $\mathbf{y}(t)$ was already low at the beginning of that pulse (i.e., $\mathbf{y}(\mathbf{t}'_{-1}) = 0$). This corresponds to the filter arriving at state S_0 at t_0^- via either S_1 or S_2, and can be formally expressed as :

$$\begin{aligned}\mathcal{P}\{\mathbf{y}(t_0) = 0 \mid \mathbf{x}(t_0) = \uparrow\} &= \mathcal{P}\left\{t_0 - \mathbf{t}'_{-1} > \tau_0 \text{ or } \mathbf{y}(\mathbf{t}'_{-1}) = 0 \mid \mathbf{x}(t_0) = \uparrow\right\} \\ &= \mathcal{P}\left\{t_0 - \mathbf{t}'_{-1} > \tau_0 \mid \mathbf{x}(t_0) = \uparrow\right\} + \mathcal{P}\left\{\mathbf{y}(\mathbf{t}'_{-1}) = 0 \mid \mathbf{x}(t_0) = \uparrow\right\} \\ &\quad - \mathcal{P}\left\{t_0 - \mathbf{t}'_{-1} > \tau_0,\ \mathbf{y}(\mathbf{t}'_{-1}) = 0 \mid \mathbf{x}(t_0) = \uparrow\right\} \qquad (B.3)\end{aligned}$$

Recall that each of the pulse widths of $\mathbf{x}(t)$ before $\mathbf{t}'_{-1}$ is independent of $t_0 - \mathbf{t}'_{-1}$. Since $\mathbf{y}(\mathbf{t}'_{-1})$ is completely determined by that (infinite) sequence of pulses, it follows that $\mathbf{y}(\mathbf{t}'_{-1})$ and $t_0 - \mathbf{t}'_{-1}$ are independent. Therefore :

$$\mathcal{P}\left\{\mathbf{y}(t_0) = 0 \mid \mathbf{x}(t_0) = \uparrow\right\} = 1 - F_0(\tau_0) + F_0(\tau_0)\mathcal{P}\left\{\mathbf{y}(\mathbf{t}'_{-1}) = 0 \mid \mathbf{x}(t_0) = \uparrow\right\} \tag{B.4}$$

Using similar arguments, we can write :

$$\begin{aligned}\mathcal{P}\left\{\mathbf{y}(\mathbf{t}'_{-1}) = 0 \mid \mathbf{x}(t_0) = \uparrow\right\} &= \mathcal{P}\left\{\mathbf{t}'_{-1} - \mathbf{t}_{-1} \le \tau_1,\ \mathbf{y}(\mathbf{t}_{-1}) = 0 \mid \mathbf{x}(t_0) = \uparrow\right\} \\ &= F_1(\tau_1)\mathcal{P}\left\{\mathbf{y}(\mathbf{t}_{-1}) = 0 \mid \mathbf{x}(t_0) = \uparrow\right\}\end{aligned} \tag{B.5}$$

where we have used the fact that, by the same argument given for $\mathbf{t}'_{-1}$, $\mathbf{y}(\mathbf{t}_{-1})$ and $\mathbf{t}'_{-1} - \mathbf{t}_{-1}$ are independent (in fact, we also have that $\mathbf{y}(\mathbf{t}_{-1})$ and $\mathbf{t}_{-1}$ are independent). Therefore :

$$\mathcal{P}\left\{\mathbf{y}(t_0) = 0 \mid \mathbf{x}(t_0) = \uparrow\right\} = 1 - F_0(\tau_0) + F_0(\tau_0)F_1(\tau_1)\mathcal{P}\left\{\mathbf{y}(\mathbf{t}_{-1}) = 0 \mid \mathbf{x}(t_0) = \uparrow\right\} \tag{B.6}$$

Since $\mathbf{y}(\mathbf{t}_{-1})$ and $\mathbf{t}_{-1}$ are independent, then for any fixed $t^* < t_0$ we have :

$$\mathcal{P}\left\{\mathbf{y}(\mathbf{t}_{-1}) = 0 \mid \mathbf{x}(t_0) = \uparrow\right\} = \mathcal{P}\left\{\mathbf{y}(\mathbf{t}_{-1}) = 0 \mid \mathbf{t}_{-1} = t^*, \mathbf{x}(t_0) = \uparrow\right\} \tag{B.7}$$

The event $\{\mathbf{t}_{-1} = t^*\}$ is equivalent to (the intersection of) the two events $\{\mathbf{x}(t^*) = \uparrow\}$ and $\{\mathbf{x}(t)$ makes no $0 \to 1$ transitions in the interval $(t^*, t_0)\}$. However, when $\mathbf{x}(t^*) = \uparrow$, $\mathbf{y}(\mathbf{t}_{-1}) = \mathbf{y}(t^*)$ is independent of $\mathbf{x}(t)$ for all time larger than t^*. Therefore :

$$\mathcal{P}\left\{\mathbf{y}(\mathbf{t}_{-1}) = 0 \mid \mathbf{x}(t_0) = \uparrow\right\} = \mathcal{P}\left\{\mathbf{y}(t^*) = 0 \mid \mathbf{x}(t^*) = \uparrow\right\} = \mathcal{P}\left\{\mathbf{y}(t_0) = 0 \mid \mathbf{x}(t_0) = \uparrow\right\} \tag{B.8}$$

where we have used (B.2) to write the last equality. This, coupled with (B.6), leads to (B.1) and completes the proof. ■

Likewise, one can show that :

$$\mathcal{P}\left\{\mathbf{y}(t_0) = 1 \mid \mathbf{x}(t_0) = \downarrow\right\} = \frac{1 - F_1(\tau_1)}{1 - F_0(\tau_0)F_1(\tau_1)} \tag{B.9}$$

B.1. Equilibrium probability

We will make use of the following lemma that gives the distribution of the time-to-transition of $\mathbf{x}(t)$, given that $\mathbf{x}(t_0)$ is known.

Lemma 2. *If $\mathbf{x}(t_0) = 0$ at some fixed time t_0, and $\mathbf{t}'_0$ is the (random) time of the last $1 \to 0$ transition before t_0, then the cdf of $t_0 - \mathbf{t}'_0$ is given by :*

$$\mathcal{P}\{t_0 - \mathbf{t}'_0 \le t\} = \hat{F}_0(t) + \frac{t}{\mu_0}[1 - F_0(t)] \tag{B.10}$$

Proof : Consider again the situation in Fig. 2, where $\mathbf{x}(t_0) = 0$ at some fixed time t_0 and $\mathbf{t_0'}$ is the (random) time of the last $1 \rightarrow 0$ transition before t_0. Using the same notation as in Fig. 2, let $\hat{\mathbf{r}}_\mathbf{0}$ denote the width of the low pulse around t_0. If $\hat{\mathbf{r}}_\mathbf{0}$ is shorter than t, then $t_0 - \mathbf{t_0'}$ is also shorter than t. Therefore :

$$\begin{aligned} \mathcal{P}\{t_0 - \mathbf{t_0'} \leq t\} &= \mathcal{P}\{\hat{\mathbf{r}}_\mathbf{0} \leq t\} + \mathcal{P}\{t_0 - \mathbf{t_0'} \leq t \mid \hat{\mathbf{r}}_\mathbf{0} > t\}\mathcal{P}\{\hat{\mathbf{r}}_\mathbf{0} > t\} \\ &= \hat{F}_0(t) + [1 - \hat{F}_0(t)]\ \mathcal{P}\{t_0 - \mathbf{t_0'} \leq t \mid \hat{\mathbf{r}}_\mathbf{0} > t\} \end{aligned} \qquad (B.11)$$

where (recall) $\hat{F}_0(t)$ is the cdf of $\hat{\mathbf{r}}_\mathbf{0}$. Let $n_{x,0}^{t+}(T)$ denote the number of low pulses of $x(t)$ in $(\frac{-T}{2}, \frac{+T}{2}]$ that are longer than t.

Let $\mathcal{R}_0^{t+}$ be the set of all $x = 0$ time intervals that are longer than t, and $\mathcal{R}_0^t \subseteq \mathcal{R}_0^{t+}$ be the set of all initial segments of length t of every interval in $\mathcal{R}_0^{t+}$. Therefore :

$$\mathcal{P}\{t_0 - \mathbf{t_0'} \leq t \mid \hat{\mathbf{r}}_\mathbf{0} > t\} = \mathcal{P}\{t_0 + \boldsymbol{\tau} \in \mathcal{R}_0^t \mid t_0 + \boldsymbol{\tau} \in \mathcal{R}_0^{t+}\} = \frac{\mathcal{P}\{\boldsymbol{\tau} \in \mathcal{R}_0^t\}}{\mathcal{P}\{\boldsymbol{\tau} \in \mathcal{R}_0^{t+}\}} \qquad (B.12)$$

where the last equality follows because $t_0 + \boldsymbol{\tau}$ has the same distribution as $\boldsymbol{\tau}$ [1] and $\mathcal{R}_0^t \subseteq \mathcal{R}_0^{t+}$. If $\boldsymbol{\tau}_T$ is uniform over $(\frac{-T}{2}, \frac{+T}{2}]$ with the cdf $F_{\boldsymbol{\tau}_T}(t) = 1/T$, for $t \in (\frac{-T}{2}, \frac{+T}{2}]$, then :

$$\mathcal{P}\{\boldsymbol{\tau} \in \mathcal{R}_0^t\} = \lim_{T\to\infty} \mathcal{P}\{\boldsymbol{\tau}_T \in \mathcal{R}_0^t\} = \lim_{T\to\infty} \frac{t n_{x,0}^{t+}(T)}{T} \qquad (B.13)$$

and :

$$\mathcal{P}\{\boldsymbol{\tau} \in \mathcal{R}_0^{t+}\} = \lim_{T\to\infty} \mathcal{P}\{\boldsymbol{\tau}_T \in \mathcal{R}_0^{t+}\} = \lim_{T\to\infty} \frac{E[\mathbf{r_0} \mid \mathbf{r_0} > t] n_{x,0}^{t+}(T)}{T} \qquad (B.14)$$

where (recall) $\mathbf{r_0}$ is distributed as the low pulses of $x(t)$. This leads to :

$$\mathcal{P}\{t_0 - \mathbf{t_0'} \leq t \mid \hat{\mathbf{r}}_\mathbf{0} > t\} = \lim_{T\to\infty} \frac{t n_{x,0}^{t+}(T)}{E[\mathbf{r_0} \mid \mathbf{r_0} > t] n_{x,0}^{t+}(T)} = \frac{t}{E[\mathbf{r_0} \mid \mathbf{r_0} > t]} \qquad (B.15)$$

It can be shown that $E[\mathbf{r_0} \mid \mathbf{r_0} > \ t] = \int_t^\infty z f(z) dz / (1 - F_0(t)) = \mu_0[1 - \hat{F}_0(t)]/[1 - F_0(t)]$, which leads to :

$$\mathcal{P}\{t_0 - \mathbf{t_0'} \leq t\} = \hat{F}_0(t) + \frac{t}{\mu_0}[1 - F_0(t)] \qquad (B.16)$$

This completes the proof. ■

Likewise, if $\mathbf{x}(t_0) = 1$ and $\mathbf{t_0'}$ is the (random) time of the last $0 \rightarrow 1$ transition before t_0, then :

$$\mathcal{P}\{t_0 - \mathbf{t_0'} \leq t\} = \hat{F}_1(t) + \frac{t}{\mu_1}[1 - F_1(t)] \qquad (B.17)$$

We are now ready to derive the expression for the filter output probability. If t_0 is any specific (fixed) time, then from basic probability theory, we have :

$$\begin{aligned} P(y) &= \mathcal{P}\{\mathbf{y}(t_0) = 1\} \\ &= \mathcal{P}\{\mathbf{y}(t_0) = 1 \mid \mathbf{x}(t_0) = 1\}\mathcal{P}\{\mathbf{x}(t_0) = 1\} \\ &\qquad + \mathcal{P}\{\mathbf{y}(t_0) = 1 \mid \mathbf{x}(t_0) = 0\}\mathcal{P}\{\mathbf{x}(t_0) = 0\} \end{aligned} \qquad (B.18)$$

If y is 1 at a time when x is 0, then y must have been 1 at the time of the last $1 \to 0$ transition of x *and* that transition must have occurred no earlier than τ_0 time units ago. Formally, this can be expressed as :

$$\mathcal{P}\{\mathbf{y}(t_0) = 1 \mid \mathbf{x}(t_0) = 0\} = \mathcal{P}\{\mathbf{y}(\mathbf{t_0'}) = 1,\ t_0 - \mathbf{t_0'} \leq \tau_0\} \tag{B.19}$$

where $\mathbf{t_0'}$ is the (random) time of the last $1 \to 0$ transition of $\mathbf{x}$ before time t_0, given that $\mathbf{x}(t_0) = 0$, as shown in Fig. 2.

With $\mathbf{x}(t_0) = 0$, $\mathbf{y}(\mathbf{t_0'})$ is completely determined by the (infinite) sequence of pulses of $\mathbf{x}(t)$ prior to $\mathbf{t_0'}$. Since all of those pulses are independent of the signal future, i.e. of $t_0 - \mathbf{t_0'}$, it follows that $\mathbf{y}(\mathbf{t_0'})$ is independent of $t_0 - \mathbf{t_0'}$, which gives :

$$\mathcal{P}\{\mathbf{y}(t_0) = 1 \mid \mathbf{x}(t_0) = 0\} = \mathcal{P}\{\mathbf{y}(\mathbf{t_0'}) = 1\}\mathcal{P}\{t_0 - \mathbf{t_0'} \leq \tau_0\} \tag{B.20}$$

Since $\mathbf{y}(\mathbf{t_0'})$ is independent of $\mathbf{t_0'}$, then $\mathcal{P}\{\mathbf{y}(\mathbf{t_0'}) = 1\} = \mathcal{P}\{\mathbf{y}(\mathbf{t_0'}) = 1 \mid \mathbf{t_0'} = t^*\}$, for some fixed $t^* < t_0$. The event $\mathbf{t_0'} = t^*$ is equal to the (intersection) of the two events $\{\mathbf{x}(t^*) =\downarrow\}$ (i.e. $\mathbf{x}(t)$ makes a $1 \to 0$ transition at t^*) and $\{\mathbf{x}(t) = 0$ over the interval $[t^*, t_0]\}$. However, $\mathbf{x}(t^*) =\downarrow$ makes $\mathbf{y}(t^*)$ independent of the future of $\mathbf{x}(t)$ after t^*, leading to simply $\mathcal{P}\{\mathbf{y}(\mathbf{t_0'}) = 1\} = \mathcal{P}\{\mathbf{y}(t^*) = 1 \mid \mathbf{x}(t^*) =\downarrow\}$. The expressions for this and for $\mathcal{P}\{t_0 - \mathbf{t_0'} \leq \tau_0\}$ are given by lemmas 1 and 2, and lead to :

$$\mathcal{P}\{\mathbf{y}(t_0) = 1 \mid \mathbf{x}(t_0) = 0\} = \left\{\hat{F}_0(\tau_0) + \frac{\tau_0}{\mu_0}[1 - F_0(\tau_0)]\right\} \frac{[1 - F_1(\tau_1)]}{[1 - F_0(\tau_0)F_1(\tau_1)]} \tag{B.21}$$

Similarly, if $\mathbf{x}(t_0) = 1$ and $\mathbf{t_0'}$ is (re)defined as the last $0 \to 1$ transition time before t_0, then :

$$\mathcal{P}\{\mathbf{y}(t_0) = 1 \mid \mathbf{x}(t_0) = 1\} = 1 - \mathcal{P}\{\mathbf{y}(\mathbf{t_0'}) = 0\}\mathcal{P}\{t_0 - \mathbf{t_0'} \leq \tau_1\} \tag{B.22}$$

which gives :

$$\mathcal{P}\{\mathbf{y}(t_0) = 1 \mid \mathbf{x}(t_0) = 1\} = 1 - \left\{\hat{F}_1(\tau_1) + \frac{\tau_1}{\mu_1}[1 - F_1(\tau_1)]\right\} \frac{[1 - F_0(\tau_0)]}{[1 - F_0(\tau_0)F_1(\tau_1)]} \tag{B.23}$$

Consolidating all this gives the required expression for the filter output probability :

$$\begin{aligned} P(y) = P(x) &- \left\{\hat{F}_1(\tau_1) + \frac{\tau_1}{\mu_1}[1 - F_1(\tau_1)]\right\} \frac{[1 - F_0(\tau_0)]}{[1 - F_0(\tau_0)F_1(\tau_1)]}P(x) \\ &+ \left\{\hat{F}_0(\tau_0) + \frac{\tau_0}{\mu_0}[1 - F_0(\tau_0)]\right\} \frac{[1 - F_1(\tau_1)]}{[1 - F_0(\tau_0)F_1(\tau_1)]}[1 - P(x)] \end{aligned} \tag{B.24}$$

B.2. Transition density

If $D(x) = 0$ then $D(y) = 0$; otherwise, the ratio $D(y)/D(x) = \lim_{T\to\infty} n_y(T)/n_x(T)$ exists. We will first show that this ratio is equal to the probability $\mathcal{P}\{\mathbf{y}(t_0 + \tau_1) =\uparrow \mid \mathbf{x}(t_0) =\uparrow\}$, for any fixed time t_0.

Let $n_{x\uparrow}(T)$ denote the number of $0 \rightarrow 1$ transitions of $x(t)$ in $(\frac{-T}{2}, \frac{+T}{2}]$. Similarly define $n_{y\uparrow}(T)$. Since $|n_x(T) - 2n_{x\uparrow}(T)| \leq 1$ and $|n_y(T) - 2n_{y\uparrow}(T)| \leq 1$, then :

$$D(x) = 2 \lim_{T \to \infty} \frac{n_{x\uparrow}(T)}{T} \qquad \text{and} \qquad D(y) = 2 \lim_{T \to \infty} \frac{n_{y\uparrow}(T)}{T} \tag{B.25}$$

Note that $0 \rightarrow 1$ $(1 \rightarrow 0)$ transitions at $\mathbf{y}$ are in one-to-one correspondence with $0 \rightarrow 1$ $(1 \rightarrow 0)$ transitions at $\mathbf{x}$ that get *transmitted* to $\mathbf{y}$. If $\mathcal{R}_{x\uparrow} \triangleq \{t : x(t) = \uparrow\}$ is the set of transition time points of $x(t)$, and $\mathcal{R}_{y\uparrow} \triangleq \{t : y(t + \tau_1) = \uparrow\}$, then $\mathcal{R}_{y\uparrow} \subseteq \mathcal{R}_{x\uparrow}$ and :

$$\begin{aligned} \mathcal{P}\{\mathbf{y}(t_0 + \tau_1) = \uparrow \ \mid \mathbf{x}(t_0) = \uparrow\} &= \mathcal{P}\{t_0 + \boldsymbol{\tau} \in \mathcal{R}_{y\uparrow} \mid t_0 + \boldsymbol{\tau} \in \mathcal{R}_{x\uparrow}\} \\ &= \lim_{T \to \infty} \mathcal{P}\{\boldsymbol{\tau}_T \in \mathcal{R}_{y\uparrow} \mid \boldsymbol{\tau}_T \in \mathcal{R}_{x\uparrow}\} \end{aligned} \tag{B.26}$$

where we have made use of the fact that $\boldsymbol{\tau}$ has the same distribution as $\boldsymbol{\tau} + t_0$ and that $\boldsymbol{\tau}_T$ is uniform on $(\frac{-T}{2}, \frac{+T}{2}]$. If $n_{y\uparrow}(\frac{-T}{2} + \tau_1, \frac{+T}{2} + \tau_1)$ denotes the number of $0 \rightarrow 1$ transitions of $y(t)$ in $(\frac{-T}{2} + \tau_1, \frac{+T}{2} + \tau_1]$, then (B.26) gives :

$$\mathcal{P}\{\mathbf{y}(t_0 + \tau_1) = \uparrow \ \mid \mathbf{x}(t_0) = \uparrow\} = \lim_{T \to \infty} \frac{n_{y\uparrow}(\frac{-T}{2} + \tau_1, \frac{+T}{2} + \tau_1)}{n_{x\uparrow}(T)} \tag{B.27}$$

Notice that $n_{y\uparrow}(T - 2\tau_1) \leq n_{y\uparrow}(\frac{-T}{2} + \tau_1, \frac{+T}{2} + \tau_1) \leq n_{y\uparrow}(T + 2\tau_1)$, so that :

$$\frac{n_{y\uparrow}(T - 2\tau_1)}{n_{x\uparrow}(T)} \leq \frac{n_{y\uparrow}(\frac{-T}{2} + \tau_1, \frac{+T}{2} + \tau_1)}{n_{x\uparrow}(T)} \leq \frac{n_{y\uparrow}(T + 2\tau_1)}{n_{x\uparrow}(T)} \tag{B.28}$$

As $T \rightarrow \infty$, the first and last terms of this inequality converge to $D(y)/D(x)$, since :

$$\lim_{T \to \infty} \frac{n_{y\uparrow}(T \pm 2\tau_1)}{n_{x\uparrow}(T)} = \lim_{T \to \infty} \left\{ \frac{n_{y\uparrow}(T \pm 2\tau_1)}{T \pm 2\tau_1} \times \frac{T \pm 2\tau_1}{T} \times \frac{T}{n_{x\uparrow}(T)} \right\} = \frac{D(y)}{D(x)} \lim_{T \to \infty} \frac{T \pm 2\tau_1}{T} \tag{B.29}$$

Therefore :

$$\frac{D(y)}{D(x)} = \mathcal{P}\{\mathbf{y}(t_0 + \tau_1) = \uparrow \ \mid \mathbf{x}(t_0) = \uparrow\} \tag{B.30}$$

We will now derive an expression for the probability in (B.30). A $0 \rightarrow 1$ transition at $\mathbf{x}$ gets transmitted (after a delay of τ_1) to $\mathbf{y}$ if and only if it takes the filter from state S_0 to state S_2 *and* is then followed by a high pulse that is longer than τ_1. Therefore, if $\mathbf{x}(t_0) = \uparrow$ and $\mathbf{t_0'} > t_0$ is the (random) time of the first $1 \rightarrow 0$ transition of $\mathbf{x}(t)$ after t_0, then the probability that the transition at t_0 is transmitted to $\mathbf{y}$ is given by :

$$\begin{aligned} \mathcal{P}\{\mathbf{y}(t_0 + \tau_1) = \uparrow \ \mid \mathbf{x}(t_0) = \uparrow\} &= \mathcal{P}\left\{\mathbf{y}(t_0) = 0, \mathbf{t_0'} - t_0 > \tau_1 \mid \mathbf{x}(t_0) = \uparrow\right\} \\ &= \mathcal{P}\{\mathbf{y}(t_0) = 0 \mid \mathbf{x}(t_0) = \uparrow\} \, \mathcal{P}\left\{\mathbf{t_0'} - t_0 > \tau_1 \mid \mathbf{x}(t_0) = \uparrow\right\} \end{aligned} \tag{B.31}$$

where we have used the two facts : (1) $\mathbf{t_0'} - t_0$ is independent of all occurrences before t_0, and (2) $\mathbf{y}(t_0)$ is completely determined by occurrences before t_0, to conclude that the two events $\{\mathbf{y}(t_0) = 0\}$ and $\{\mathbf{t_0'} - t_0 > \tau_1\}$ are independent.

The second term in (B.31) is simply : $\mathcal{P}\{\mathbf{t_0'} - t_0 > \tau_1 \mid \mathbf{x}(t_0) = \uparrow\} = 1 - F_1(\tau_1)$. As for the first term, $\mathcal{P}\{\mathbf{y}(t_0) = 0 \mid \mathbf{x}(t_0) = \uparrow\}$, its value is derived by lemma 1, leading to :

$$\mathcal{P}\left\{\mathbf{y}(t_0 + \tau_1) = \uparrow \ \mid \mathbf{x}(t_0) = \uparrow\right\} = \frac{[1 - F_0(\tau_0)]\,[1 - F_1(\tau_1)]}{1 - F_0(\tau_0) F_1(\tau_1)} \qquad (B.32)$$

so that the density at the filter output is given by :

$$D(y) = \frac{[1 - F_0(\tau_0)]\,[1 - F_1(\tau_1)]}{1 - F_0(\tau_0) F_1(\tau_1)} D(x) \qquad (B.33)$$

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References

[1] F. Najm, "Transition density: a new measure of activity in digital circuits," *IEEE Transactions on Computer-Aided Design*, , vol. 12, no. 2, pp. 310-323, February 1993

[2] A. Papoulis, *Probability, Random Variables, and Stochastic Processes*, 2nd Edition. New York, NY: McGraw-Hill Book Co., 1984.

[3] E. Parzen, *Stochastic Processes*, San Francisco, CA: Holden-Day Inc., 1962.

[4] D. R. Cox & H. D. Miller, *The Theory of Stochastic Processes*, New York: John Wiley & Sons Inc., 1968.

[5] S. M. Ross, *Stochastic Processes*, New York: John Wiley & Sons Inc., 1983.